Space-Time Current Process for Independent Random Walks in One Dimension

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Abstract

In a system made up of independent random walks, fluctuations of order $n^{1/4}$ from the hydrodynamic limit come from particle current across characteristics. We show that a two-parameter space-time particle current process converges to a two-parameter Gaussian process. These Gaussian processes also appear as the limit for the one-dimensional random average process. The final section of this paper looks at large deviations of the current process.

Key words. Independent random walks, hydrodynamic limit, fluctuations, large deviation.

AMS subject classifications. Primary 60K35, 60F10; secondary 60F17, 60G15.

1 Introduction

It is well known that particle systems that appear different at the microscopic level often behave almost identically at a macroscopic level. This has been observed in the hydrodynamic limits and fluctuation results of several particle models. Consequently, there is much to be gained in studying the behavior of simpler stochastic particle systems in the hope that at the macroscopic level they will reflect the behavior of a universal class of systems. While the hydrodynamic limit of several models have been studied, fluctuation results have proved elusive for many systems. In this paper we consider particle current fluctuations in the one dimensional independent random walk model.

The hydrodynamic limit for particle distribution in typical asymmetric systems are solutions to p.d.e's of the form

$$\partial_t u + \partial_x f(u) = 0. \tag{1.1}$$

In the case of nearest neighbor Totally Asymmetric Simple Exclusion Process (TASEP) in one dimension, the flux function $f(\rho) = \rho(1-\rho)$. For non-interacting particle systems $f(\rho) = v \cdot \rho$ where v is the average velocity of the particles. Thus the relevant p.d.e. for a system of independent asymmetric random walks is

$$\partial_t u + v \cdot \partial_x u = 0 \tag{1.2}$$

(Prop 3.1, page 15 in [8]).

From the transport equation (1.2) we see that in the independent random walk model, the initial density profile shifts with velocity v. Consider an observer moving at constant velocity v. The path of the observer is a characteristic line of the transport equation (1.2). It is natural to expect the net current of particles across the path of the observer to be zero. But what are the fluctuations in this particle current? This is the question we address in this paper. It has been observed that these current fluctuations are of order $n^{1/4}$ [13]. Here, n is the scaling parameter. Typically we scale both space and time by n in asymmetric models, this is called Euler scaling. In symmetric models we use diffusive scaling i.e. we scale time by n and space by $\sqrt{n}$. There is a general belief that when f is linear (i.e. $f'' \equiv 0$) in (1.1), the fluctuations in particle current across characteristics of (1.1) should be of order $n^{1/4}$. This has been shown for the random average process (RAP) and for the one dimensional independent random walk model where $f'' \equiv 0$.

In this paper we study both the fluctuations and the large deviations of the current process for independent walks. For the fluctuations we consider the current process indexed both by time and spatial shifts of order $\sqrt{n}$ of characteristic lines. The $\sqrt{n}$ order for spatial scaling is the natural one because the individual random walks fluctuate on that scale. We extend the distributional limit of [13] to a process limit for the space-time current process. The space-time current process was also studied for RAP in [1] but only convergence of finite dimensional distributions was shown without process-level tightness. The same family of Gaussian processes arises as limits for both RAP and independent random walks.

It is interesting to note that there are models which are not asymmetric yet exhibit subdiffusive current fluctuations with Gaussian scaling limits. It was conjectured (conjecture 6.5 in [14]) that subdiffusive fluctuations in 1 dimensional nearest neighbor symmetric simple exclusion processes (SSEP) converge to fBM with Hurst parameter 1/4. It was subsequently proved in the finite dimensional distributions sense and has recently been proved in the full functional central limit theorem sense in [11]. This says that the universality class of current fluctuations of order $n^{1/4}$ contains both symmetric and asymmetric processes. However, the symmetric and asymmetric processes differ on the level of hydrodynamics.

This paper is organized as follows. We start with a description of the independent random walk model and the statements of the main results in section 2. The next three sections 3, 4 and 5 cover the proofs. Section 3 gives the convergence of finite dimensional distributions and section 4 proves process level tightness. A note on the tightness methods used here: since we are interested in a two-parameter process, the standard theorems on convergence in $D_{\mathbb{R}}[0,\infty)$ and $C_{\mathbb{R}}[0,\infty)$ spaces do not apply. We appeal to two papers, [2] and [6], that provide suitable criteria for deducing tightness. [2] gives the context in which we speak of convergence for two-parameter processes and a tightness criterion. The proof of Proposition 5.7 in [6] is extended to two dimensions to prove the tightness criterion. The last section contains proofs of some large deviation results for the current process.

2 Model and results

2.1 Independent random walk model

Consider particles distributed over the one dimensional integer lattice which evolve like independent continuous-time random walks. We are given the initial occupation variables $\eta_0 = \{\eta_0(x) : x \in \mathbb{Z}\}$ defined on some probability space. Let $X_{m,j}(t)$ denote the position at time t of the jth random walk starting at site m. The common jump rates of the random walks are given by a probability kernel $\{p(x) : x \in \mathbb{Z}\}$. Once the initial positions of the random walks are specified, their subsequent evolutions $\{X_{m,j}(t) - X_{m,j}(0) : m \in \mathbb{Z}, j = 1, \ldots, \eta_0(m)\}$ are as i.i.d. random walks starting at the origin, on the same probability space, independent of η_0. Define

$$\eta_t(x) := \sum_{m\in\mathbb{Z}} \sum_{j=1}^{\eta_0(m)} \mathbf{1}\{X_{m,j}(t) = x\}$$

to be number of particles on site x at time t.

Assumption 2.1. *For the random walk kernel, we assume that, for some $\delta > 0$,*

$$\sum_{x\in\mathbb{Z}} e^{\theta x} p(x) < \infty \text{ for } |\theta| \leq \delta \tag{2.1}$$

(This assumption will enable us to calculate large deviation bounds for the random walks.)

Throughout this paper, we assume that $\mathbb{N}$ denotes the set of positive integers. Let $\{\eta_0^n : n \in \mathbb{N}\}$ be a sequence of initial occupation variables defined on some probability space.

Assumption 2.2. *For each n, the initial occupation variables $\{\eta_0^n(x) : x \in \mathbb{Z}\}$ are independent. They have a uniformly bounded twelfth moment:*

$$\sup_{n \in N, x \in \mathbb{Z}} E[\eta_0^n(x)^{12}] < \infty. \tag{2.2}$$

Let $\rho_0^n(x) = E\eta_0^n(x)$ and $v_0^n(x) = Var[\eta_0^n(x)]$ be the mean and variance resp. of the initial occupation variable $\eta_0^n(x), x \in \mathbb{Z}$. Let ρ_0 and v_0 be two given nonnegative, finite numbers. The means ρ_0^n and variances v_0^n approximate ρ_0 and v_0 in the following sense: There exist positive integers $L = L(n)$ such that $n^{-1/4}L(n) \to 0$ and for any finite constant A,

$$\lim_{n\to\infty} \sup_{|m| \le A\sqrt{n \log n}} n^{1/4} \Big| \frac{1}{L(n)} \sum_{j=1}^{L(n)} \rho_0^n(m+j) - \rho_0 \Big| = 0 \tag{2.3}$$

The same assumption holds when ρ_0^n and ρ_0 are replaced by v_0^n and v_0.

As in [13], the reason for the complicated assumption (2.3) is to accommodate both random and deterministic initial conditions. For random $\eta_0^n(x)$ we could take $\rho_0^n(x) = \rho_0(\frac{x}{n})$ for some sufficiently regular function $\rho_0(\cdot)$. However, for deterministic $\eta_0^n(x)$ we cannot do this unless $\rho_0(x)$ is integer-valued. A couple of examples illustrating random and deterministic initial configurations that satisfy assumptions 2.1 and 2.2 can be found in [13].

Let

$$v = \sum_x xp(x) \text{ and } \kappa_2 = \sum_x x^2 p(x).$$

The characteristics of (1.2) are straight lines with slope v. Fix $T > 0$ and $S > 0$. For $t \in [0,T]$ and $r \in [-S,S]$, we let $Y_n(t,r)$ denote the net right-to-left particle current during time $[0,nt]$ across the characteristic line starting at $([r\sqrt{n}],0)$. More precisely,

$$\begin{aligned} Y_n(t,r) := \sum_{m=-\infty}^{\infty} \sum_{j=1}^{\eta_0^n(m)} \big[&\mathbf{1}\{X_{m,j}(nt) \le [nvt] + [r\sqrt{n}]\}\mathbf{1}\{m > [r\sqrt{n}]\} \\ &- \mathbf{1}\{X_{m,j}(nt) > [nvt] + [r\sqrt{n}]\}\mathbf{1}\{m \le [r\sqrt{n}]\}\big] \end{aligned} \tag{2.4}$$

where $X_{k,j}(\cdot)$ is the jth random walk that starts at site k. Note that the random walks denoted as $X_{k,j}$ in the definition of $Y_n(t,r)$ should actually be $X_{k,j}^n$, but we drop the superscript n for notational simplicity.

2.2 Distributional limit

We give a brief description of the path space of the process $Y_n(\cdot,\cdot)$. Let $D_2 = D_2([0,T] \times [-S,S], \mathbb{R})$ be the space of 2-parameter cadlag functions with Skorohod's topology. Let

$Q := [0,T] \times [-S,S]$. For any $(t,r) \in Q$, we can divide Q into four quadrants:

$$Q^1_{(t,r)} := \{(s,q) \in Q : s \geq t, q \geq r\},\ Q^2_{(t,r)} := \{(s,q) \in Q : s \geq t, q < r\},$$

$$Q^3_{(t,r)} := \{(s,q) \in Q : s < t, q < r\},\ Q^4_{(t,r)} := \{(s,q) \in Q : s < t, q \geq r\}.$$

Then the precise definition of D_2 is

$$D_2 = \{f : Q \to \mathbb{R} \text{ such that for any point } (t,r) \in Q, \lim_{\substack{(s,q)\in Q^i_{(t,r)} \\ (s,q)\to(t,r)}} f(s,q)$$

$$\text{exists for } i = 1,2,3,4 \text{ and } \lim_{\substack{(s,q)\in Q^1_{(t,r)} \\ (s,q)\to(t,r)}} f(s,q) = f(t,r)\}.$$

In other words, D_2 contains functions that are continuous from the right and above with limits from the left and below. Skorohod's topology in $D_{\mathbb{R}}[0,\infty)$ is extended to this space in the most natural way. The space of multiparameter cadlag functions and their topology is described in detail in [2]. By Theorem 2 in [2], a sufficient criterion for the weak convergence $X_n \to X$ in D_2 is,

1. For all finite subsets $\{(t_i,r_i)\} \subset [0,T] \times [-S,S]$,

$$(X_n(t_1,r_1),\cdots,X_n(t_N,r_N)) \to (X(t_1,r_1),\cdots,X(t_N,r_N))$$

   weakly, and

2. $\lim_{\delta\to 0} \limsup_n P\{w_{X_n}(\delta) \geq \epsilon\} = 0$ for all $\epsilon > 0$, where the modulus of continuity is defined by

$$w_x(\delta) = \sup_{\substack{(s,q),(t,r)\in[0,T]\times[-S,S] \\ |(s,q)-(t,r)|<\delta}} |x(s,q) - x(t,r)|.$$

Clearly, $\{Y_n(t,r) : t \in [0,T], r \in [-S,S]\}$ are D_2-valued processes and we can use the above criterion to prove their convergence in the weak sense.

Denote the centered Gaussian density and distribution with variance σ^2 by

$$\phi_{\sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{1}{2\sigma^2}x^2\} \text{ and } \Phi_{\sigma^2}(x) = \int_{-\infty}^{x} \phi_{\sigma^2}(y)dy.$$

Define also

$$\Psi_{\sigma^2}(x) = \sigma^2\phi_{\sigma^2}(x) - x(1 - \Phi_{\sigma^2}(x)).$$

For $(s,q),(t,r) \in [0,T] \times [-S,S]$, define two covariances by

$$\Gamma_0((s,q),(t,r)) = \Psi_{\kappa_2 s}(|q-r|) + \Psi_{\kappa_2 t}(|q-r|) - \Psi_{\kappa_2(t+s)}(|q-r|) \tag{2.5}$$

and

$$\Gamma_q((s,q),(t,r)) = \Psi_{\kappa_2(t+s)}(|q-r|) - \Psi_{\kappa_2|t-s|}(|q-r|). \tag{2.6}$$

THEOREM 2.1. *Define $Y_n(t,r)$ as in (2.4). Then under Assumptions 2.1 and 2.2, as $n \to \infty$, the process $n^{-1/4}Y_n(\cdot,\cdot)$ converges weakly on the space D_2 to the mean zero Gaussian process $Z(\cdot,\cdot)$ with covariance*

$$EZ(s,q)Z(t,r) = \rho_0\Gamma_q((s,q),(t,r)) + v_0\Gamma_0((s,q),(t,r)). \tag{2.7}$$

Note: We will show later that $n^{-1/4}EY_n(t,r) \to 0$ as $n \to \infty$ uniformly for $t \in [0,T]$ and $r \in [-S,S]$. Hence, in the above theorem we can replace $n^{-1/4}Y_n$ with the centered current process with impunity.

COROLLARY 2.1. *Under the invariant distribution where $\{\eta_t^n(x) : x \in \mathbb{Z}\}$ are i.i.d. Poisson with mean ρ for all n, $n^{-1/4}Y_n(\cdot,\cdot)$ converges weakly in D_2 to a mean-zero Gaussian process $Z(\cdot,\cdot)$ with covariance*

$$EZ(s,q)Z(t,r) = \rho\big(\Psi_{\kappa_2 s}(|q-r|) + \Psi_{\kappa_2 t}(|q-r|) - \Psi_{\kappa_2|t-s|}(|q-r|)\big).$$

In particular, for a fixed r the process $\{Z(t,r) : t \in [0,T]\}$ has covariance

$$EZ(s,r)Z(t,r) = \rho\sqrt{\frac{\kappa_2}{2\pi}}\big(\sqrt{s} + \sqrt{t} - \sqrt{|t-s|}\big),$$

i.e., process $Z(\cdot,r)$ is fractional Brownian motion with Hurst parameter $1/4$.

The covariance (2.7) is the same as the covariance of the limiting Gaussian process in the RAP model found in [1], with different coefficients in front of Γ_q and Γ_0. As in [1], we can represent the Gaussian process $Z(\cdot,\cdot)$ as a stochastic integral:

$$\begin{aligned} Z(t,r) = \sqrt{\kappa_2\rho_0}&\int_{[0,t]\times\mathbb{R}} \phi_{\kappa_2(t-s)}(r-z)dW(s,z) \\ &+ \sqrt{v_0}\int_{\mathbb{R}} sgn(x-r)\Phi_{\kappa_2 t}(-|x-r|)dB(x). \end{aligned} \tag{2.8}$$

The equality in (2.8) is equality in distribution. W is a two-parameter Brownian motion defined on $\mathbb{R}_+ \times \mathbb{R}$, B is a one-parameter Brownian motion defined on $\mathbb{R}$, and W and B are independent of each other. The stochastic integral clearly delineates the two sources of fluctuations in the current. The first integral represents the space-time noise created by the dynamics, and the second integral represents the initial noise propagated by the evolution.

2.3 Large deviation results

We first state large deviation results for $Y_n(t,r)$ with fixed $r \in \mathbb{R}$ and $t > 0$.

Assumption 2.3. *Assume the initial occupation variables* $\{\eta_0^n(m) : m \in \mathbb{Z}\}$ *are i.i.d. Let*

$$\gamma(\theta) = \log E e^{\theta \eta_0^n(m)}$$

with effective domain

$$D_\gamma := \{\alpha \in \mathbb{R} : \gamma(\alpha) < \infty\}.$$

Assume $D_\gamma = \mathbb{R}$.

For $\lambda \in \mathbb{R}$, define

$$Z_\lambda(y) := \begin{cases} \log\{\Phi_{\kappa_2 t}(y) + e^\lambda(1 - \Phi_{\kappa_2 t}(y))\} & \text{for } y > 0 \\ \log\{e^{-\lambda}\Phi_{\kappa_2 t}(y) + 1 - \Phi_{\kappa_2 t}(y)\} & \text{for } y \leq 0 \end{cases}$$

and

$$\Lambda(\lambda) := \int_{-\infty}^{\infty} \gamma(Z_\lambda(y))dy$$

with effective domain

$$D_\Lambda := \{\alpha \in \mathbb{R} : \Lambda(\alpha) < \infty\}.$$

Λ turns out to be the limiting moment generating function of the current.

Recall

Definition 2.1. *A convex function* $\Lambda : \mathbb{R} \to (-\infty, \infty]$ *is essentially smooth if:*
a) D_Λ^o *(interior of* D_Λ*) is non-empty.*
b) $\Lambda(\cdot)$ *is differentiable throughout* D_Λ^o.
c) $\Lambda(\cdot)$ *is steep, i.e.,* $\lim_{n\to\infty} |\nabla\Lambda(\lambda_n)| = \infty$ *whenever* $\{\lambda_n\}$ *is a sequence in* D_Λ^o *converging to a boundary point of* D_Λ^o.

Let $\gamma^*(x) := \sup_{\lambda\in\mathbb{R}}\{\lambda \cdot x - \gamma(\lambda)\}$ be the convex dual of $\gamma(\cdot)$. Let

$$I(x) := \sup_{\lambda\in\mathbb{R}}\{\lambda \cdot x - \Lambda(\lambda)\}.$$

Recall the usual definition of Large Deviation Principle (LDP). The sequence of random variables $\{X_n\}$ satisfies the LDP with rate function $J(x)$ and normalization $\{\sqrt{n}\}$ if the following are satisfied:

1. For any closed set $F \subset \mathbb{R}$,

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n}} \log P\{X_n \in F\} \leq -\inf_{x\in F} J(x)$$

2. For any open set $G \subset \mathbb{R}$,

$$\liminf_{n\to\infty} \frac{1}{\sqrt{n}} \log P\{X_n \in G\} \geq -\inf_{x\in G} J(x)$$

The rate function J is said to be a good rate function if its level sets are compact.

Let $Br_p(\lambda) := \log Ee^{\lambda X}$, where $X \sim$Bernoulli(p), be the logarithmic moment generating function of Bernoulli random variables. Its convex dual is

$$Br_p^*(x) = x \log \frac{x}{p} + (1-x) \log \frac{1-x}{1-p}$$

for $0 \leq x \leq 1$. Define $\alpha(x)$ implicitly by

$$x = \Lambda'(\alpha(x)). \tag{2.9}$$

This definition is well defined for all $x \in \mathbb{R}$ since $\Lambda(\cdot)$ is strictly convex. For any $\alpha \in \mathbb{R}$, define

$$F_\alpha(y) := \frac{e^{-\alpha}\Phi_{\kappa_2 t}(y)}{1-\Phi_{\kappa_2 t}(y) + e^{-\alpha}\Phi_{\kappa_2 t}(y)} \text{ for } y \in \mathbb{R}. \tag{2.10}$$

By (2.9) we have

$$x = \int_0^\infty \gamma'(Z_{\alpha(x)}(y))(1 - F_{\alpha(x)}(y))dy - \int_{-\infty}^0 \gamma'(Z_{\alpha(x)}(y))F_{\alpha(x)}(y)dy. \tag{2.11}$$

Define

$$I_1(x) := \int_{-\infty}^\infty \gamma^* \left\{\gamma'(Z_{\alpha(x)}(y))\right\} dy \tag{2.12}$$

and

$$I_2(x) := \int_{-\infty}^\infty \gamma'(Z_{\alpha(x)}(y)) Br^*_{\Phi_{\kappa_2 t}(y)}(F_{\alpha(x)}(y))dy. \tag{2.13}$$

Recall that we assumed $\eta_0^n(\cdot)$ are i.i.d. at the beginning of this section. Consequently, the underlying distribution is shift invariant. The rate function in Theorem 2.2 therefore does not depend on r, as the marginals of the current process, $Y_n(t,r)$, are shift invariant.

THEOREM 2.2. *Let Assumptions 2.1 and 2.3 hold. For fixed real r and $t > 0$, $n^{-1/2}Y_n(t,r)$ satisfies the large deviation principle with normalization $\{\sqrt{n}\}$ and good, strictly convex rate function*

$$I(x) = I_1(x) + I_2(x), \quad x \in \mathbb{R}. \tag{2.14}$$

A few words on the rate function. I has a unique zero at zero. The rate function I balances two costs: the cost of deviations in the initial occupation variables, given by I_1, and the cost of deviations in the probability with which particles cross the characteristic lines, given by I_2. In the macroscopic picture, $1-\Phi_{\kappa_2 t}(y)$ is the a priori probability with which particles initially at distance $y>0$ from the characteristic line cross it by time t, while a particle at distance $y \le 0$ crosses the characteristic line with probability $\Phi_{\kappa_2 t}(y)$.

An intuitive understanding of the LDP is as follows. To allow a current of size x at time t the system deviates in such a way that its behavior is governed by a new probability measure. Under this new probability measure the mean number of particles initially at site y is $\gamma'(Z_{\alpha(x)}(y))$ and the probabilities $1-\Phi_{\kappa_2 t}(y)$ and $\Phi_{\kappa_2 t}(y)$ are tilted to give new probabilities $1-F_{\alpha(x)}(y)$ and $F_{\alpha(x)}(y)$. The term $Br_{\Phi_{\kappa_2 t}(y)}(F_{\alpha(x)}(y))$ measures the cost of the deviation of the probability $1-\Phi_{\kappa_2 t}(y)$ (or $\Phi_{\kappa_2 t}(y)$) to $1-F_{\alpha(x)}(y)$ (or $F_{\alpha(x)}(y)$). The new measure depends on $\alpha(x)$ which is chosen so that the mean current under the new measure is x, this is evident from (2.11).

The modus operandi for the proof involves using non-rigorous, intuitive ideas for finding a candidate for the rate function and then checking if it is, in fact, the correct rate function.

We give explicit formulas for the two simplest cases: the stationary situation with i.i.d. Poisson occupations, and the case of deterministic initial occupations.

COROLLARY 2.2. *1. If $\eta_0^n(\cdot) \sim Poisson(\rho)$, then under assumptions 2.1 and 2.3, the rate function is*

$$I(x) = x\log\left(\frac{x\sqrt{\pi}}{\rho\sqrt{2\kappa_2 t}} + \sqrt{1+\frac{x^2\pi}{2\rho^2\kappa_2 t}}\right) - \rho\sqrt{\frac{2\kappa_2 t}{\pi}}\left(\sqrt{1+\frac{x^2\pi}{2\rho^2\kappa_2 t}}-1\right) \text{ for } x \in \mathbb{R}. \tag{2.15}$$

2. If $\eta_0^n(\cdot) \equiv 1$, then under assumption 2.1 the rate function is

$$I(x) = \int_{-\infty}^{\infty}\left[(1-F_{\alpha(x)}(y))\log\left(\tfrac{1-F_{\alpha(x)}(y)}{1-\Phi_{\kappa_2 t}(y)}\right) + F_{\alpha(x)}(y)\log\left(\tfrac{F_{\alpha(x)}(y)}{\Phi_{\kappa_2 t}(y)}\right)\right]dy$$

where $F_{\alpha(x)}(y) = \frac{e^{-\alpha(x)}\Phi_{\kappa_2 t}(y)}{1-\Phi_{\kappa_2 t}(y)+e^{-\alpha(x)}\Phi_{\kappa_2 t}(y)}$ with $\alpha(x)$ chosen so that

$$x = \int_0^{\infty}(1-F_{\alpha(x)}(y))dy - \int_{-\infty}^{0}F_{\alpha(x)}(y)dy.$$

For the process level, under the stationary distribution, we can show an abstract LDP by applying a theorem from [7]. But currently we do not have an attractive representation of the rate function. The rate function is given in terms of a variational expression in (5.14).

THEOREM 2.3. *If $\eta_0^n(\cdot) \sim Poisson(\rho)$, then under assumptions 2.1 and 2.3 the sequence of processes $\{n^{-1/2}Y_n(\cdot,0)\}$ in $D_{\mathbb{R}}[0,\infty)$ satisfies a LDP with a good rate function.*

There exist some large deviation results for independent random walk systems in the literature [9],[3],[4]. These papers essentially deal with large deviations of occupation times of sites. Lee [9] and Cox and Durrett [3] find LDP's for weighted occupation times of sites under deterministic and stationary (i.i.d. Poisson) initial distributions. Even the normalizations of the LDP's were distinct for these two cases. This is in contrast to our current large deviations in Theorem 2.2 where the normalizations for the Poisson and the deterministic case were the same.

3 Weak Convergence of Finite Dimensional Distributions

We begin the proof of Theorem 2.1 by first showing weak convergence of the finite dimensional distributions of $n^{-1/4}Y_n(\cdot,\cdot)$.

PROPOSITION 3.1. *Define $Y_n(t,r)$ as above. Then under assumptions 2.1 and 2.2 the finite dimensional distributions of the processes $\{Y_n(t,r) : (t,r) \in [0,T] \times [-S,S]\}$ converge weakly as $n \to \infty$ to the finite-dimensional distributions of the mean zero Gaussian process $\{Z(t,r) : (t,r) \in [0,T] \times [-S,S]\}$ with covariance given in (2.7).*

To prove convergence of finite dimensional distributions we use Lindeberg-Feller and check the conditions of Lindeberg-Feller by brute force. We show that the expected value of the process converges uniformly to 0 and hence we can consider the centered process when proving convergence. We also appeal to large deviations of random walks to control the contributions to the current process from distant particles.

Fix N space-time points: $(t_1,r_1),(t_2,r_2),\ldots,(t_N,r_N)$ where $(t_i,r_i) \neq (t_j,r_j)$ for $i \neq j$. We will prove that as $n \to \infty$ the vector

$$n^{-1/4}(Y_n(t_1,r_1),Y_n(t_2,r_2),...,Y_n(t_N,r_N))$$

converges in distribution to the mean-zero Gaussian random vector

$$(Z(t_1,r_1),Z(t_2,r_2),...,Z(t_N,r_N))$$

with covariance

$$EZ(t_i,r_i)Z(t_j,r_j) = \rho_0\Gamma_q((t_i,r_i),(t_j,r_j)) + v_0\Gamma_0((t_i,r_i),(t_j,r_j)).$$

Let $\theta = (\theta_1, ..., \theta_N) \in \mathbf{R}^N$ be arbitrary. Recall that $X_{k,j}(\cdot)$ is the jth random walk that starts at site k.

$$
\begin{aligned}
&n^{-1/4} \sum_{i=1}^{N} \theta_i Y_n(t_i, r_i) \\
&= n^{-1/4} \sum_{i=1}^{N} \theta_i \Bigg\{ \sum_{m=-\infty}^{\infty} \sum_{j=1}^{\eta_0^n(m)} \big(\mathbf{1}\{X_{m,j}(nt_i) - X_{m,j}(0) \le [nvt_i] + [r_i\sqrt{n}] - m\} \mathbf{1}\{m > [r_i\sqrt{n}]\} \\
&\qquad - \mathbf{1}\{X_{m,j}(nt_i) - X_{m,j}(0) > [nvt_i] + [r_i\sqrt{n}] - m\} \mathbf{1}\{m \le [r_i\sqrt{n}]\} \big) \Bigg\}
\end{aligned}
$$

Let

$$A^{t,r}_{m,j} = \{X_{m,j}(nt) - X_{m,j}(0) \le [nvt] + [r\sqrt{n}] - m\}$$

and

$$A^{t,r} = \{X(nt) \le [nvt] + [r\sqrt{n}] - m\}$$

where $X(\cdot)$ represents a random walk with rates $p(x)$ starting at the origin. The evolution of the random walks is independent of their initial occupation numbers $\eta_0^n(x)$. Define

$$
\begin{aligned}
U_m(t, r) :=& n^{-1/4} \sum_{j=1}^{\eta_0^n(m)} \big\{ \mathbf{1}\{A^{t,r}_{m,j}\} \mathbf{1}\{m > [r\sqrt{n}]\} - \mathbf{1}\{(A^{t,r}_{m,j})^c\} \mathbf{1}\{m \le [r\sqrt{n}]\} \big\} \\
&- n^{-1/4} \rho_0^n(m) \big\{ P\{A^{t,r}\} \mathbf{1}\{m > [r\sqrt{n}]\} - P\{(A^{t,r})^c\} \mathbf{1}\{m \le [r\sqrt{n}]\} \big\}
\end{aligned}
$$

and

$$\bar{U}_m := \sum_{i=1}^{N} \theta_i U_m(t_i, r_i).$$

Then we can write

$$
\begin{aligned}
n^{-1/4} \sum_{i=1}^{N} \theta_i Y_n(t_i, r_i) =& n^{-1/4} \sum_{i=1}^{N} \theta_i \{Y_n(t_i, r_i) - EY_n(t_i, r_i)\} + n^{-1/4} \sum_{i=1}^{N} \theta_i EY_n(t_i, r_i) \\
=& \sum_{m=-\infty}^{\infty} \bar{U}_m + n^{-1/4} \sum_{i=1}^{N} \theta_i EY_n(t_i, r_i).
\end{aligned}
$$

We split $\sum_{m=-\infty}^{\infty} \bar{U}_m$ into two sums S_1 and S_2 as follows. Choose $r(n)$ so that $r(n) = o(\sqrt{\log\ n})$ and $r(n) \to \infty$ slowly enough that

$$r(n)H(n^{1/8}) \to 0 \text{ as } n \to \infty,$$

where $H(M) = \sup_{n\geq 1, x\in\mathbb{Z}} E[\eta_0^n(x)^2 1\{\eta_0^n(x) \geq M\}]$. Write

$$n^{-1/4}\sum_{i=1}^{N}\theta_i\{Y_n(t_i,r_i) - EY_n(t_i,r_i)\} = \sum_{|m|\leq r(n)\sqrt{n}} \bar{U}_m + \sum_{|m|>r(n)\sqrt{n}} \bar{U}_m$$
$$=: S_1 + S_2.$$

We show that S_2 goes to 0 in L^2 and use Lindeberg-Feller for S_1 to show that it converges to a mean-zero normal distribution.

LEMMA 3.1. *$ES_2^2 \to 0$ as $n \to \infty$.*

Proof. $EU_m = 0$, so $E\bar{U}_m = 0$. S_2 is therefore a sum of independent mean zero terms $\bar{U}_m$. By Schwarz Inequality,

$$ES_2^2 = \sum_{|m|\geq [r(n)\sqrt{n}]+1} E\bar{U}_m^2 \leq \|\theta\|^2 \sum_{i=1}^{N} \sum_{|m|\geq [r(n)\sqrt{n}]+1} EU_m^2(t_i,r_i).$$

Since N is fixed, it suffices to show for fixed (t,r)

$$\lim_{n\to\infty} \sum_{m\geq [r(n)\sqrt{n}]+1}^{\infty} EU_m^2(t,r) = 0.$$

Recall that $A_{m,j}^{t,r} = \{X_{m,j}(nt) - X_{m,j}(0) \leq [nvt] + [r\sqrt{n}] - m\}$ and $A^{t,r} = \{X(nt) \leq [nvt] + [r\sqrt{n}] - m\}$. If $T_K = \sum_{i=1}^{K} Z_i$ is a random sum of i.i.d. summands Z_i independent of random K, then

$$\mathrm{Var}[T_K] = EK \cdot \mathrm{Var}\, Z_1 + (EZ_1)^2 \cdot \mathrm{Var}\, K.$$

For fixed m, take $K = \eta_0^n(m)$ and

$$Z_j = \mathbf{1}\{A_{m,j}^{t,r}\}\mathbf{1}\{m > [r\sqrt{n}]\} - \mathbf{1}\{(A_{m,j}^{t,r})^c\}\mathbf{1}\{m \leq [r\sqrt{n}]\}.$$

Then,

$$E[U_m^2(t,r)] = \mathrm{Var}\big[n^{-1/4}\sum_{j=1}^{\eta_0^n(m)} \big\{\mathbf{1}\{A_{m,j}^{t,r}\}\mathbf{1}\{m > [r\sqrt{n}]\} - \mathbf{1}\{(A_{m,j}^{t,r})^c\}\mathbf{1}\{m \leq [r\sqrt{n}]\}\big\}\big].$$

$$\begin{aligned}\mathrm{Var}\, Z_j =& P(A_{m,j}^{t,r})P((A_{m,j}^{t,r})^c)\mathbf{1}\{m > [r\sqrt{n}]\} + P(A_{m,j}^{t,r})P((A_{m,j}^{t,r})^c)\mathbf{1}\{m \leq [r\sqrt{n}]\} \\ =& P(X(nt) \leq [nvt] + [r\sqrt{n}] - m)P(X(nt) > [nvt] + [r\sqrt{n}] - m).\end{aligned}$$

$$[EZ_1]^2 = P(X(nt) \leq [nvt] + [r\sqrt{n}] - m)^2 \mathbf{1}\{m > [r\sqrt{n}]\} \\ + P(X(nt) > [nvt] + [r\sqrt{n}] - m)^2 \mathbf{1}\{m \leq [r\sqrt{n}]\}.$$

$$E[U_m^2(t,r)] = n^{-1/2}\rho_0^n(m)P(A^{t,r})P((A^{t,r})^c) + n^{-1/2}v_0^n(m)\big[P(A^{t,r})^2\mathbf{1}\{m > [r\sqrt{n}]\} \\ + P((A^{t,r})^c)^2\mathbf{1}\{m \leq [r\sqrt{n}]\}\big].$$

Using the uniform bound (2.2) on moments we get

$$E[U_m^2(t,r)] \leq n^{-1/2}C\big[P(X(nt) \leq [nvt] + [r\sqrt{n}] - m)\mathbf{1}\{m > [r\sqrt{n}]\} \\ + P(X(nt) > [nvt] + [r\sqrt{n}] - m)\mathbf{1}\{m \leq [r\sqrt{n}]\}\big].$$

By standard large deviation theory, for arbitrarily small δ, there is a constant $0 < K < \infty$ such that when $m > [r\sqrt{n}]$,

$$P(X(nt) \leq [nvt] + [r\sqrt{n}] - m) \leq \begin{cases} e^{\{-K(m-[r\sqrt{n}])^2/nt\}} & \text{if } |m - [r\sqrt{n}]| \leq nt\delta \\ e^{\{-K|m-[r\sqrt{n}]|\}} & \text{if } |m - [r\sqrt{n}]| > nt\delta \end{cases} \tag{3.1}$$

and when $m \leq [r\sqrt{n}]$,

$$P(X(nt) > [nvt] + [r\sqrt{n}] - m) \leq \begin{cases} e^{\{-K([r\sqrt{n}]-m)^2/nt\}} & \text{if } |[r\sqrt{n}] - m| \leq nt\delta \\ e^{\{-K|[r\sqrt{n}]-m|\}} & \text{if } |[r\sqrt{n}] - m| > nt\delta. \end{cases} \tag{3.2}$$

Consequently,

$$\begin{aligned} &\sum_{|m|=[r(n)\sqrt{n}]+1}^{\infty} E[U_m^2(t,r)] \\ &\leq Cn^{-1/2} \sum_{m_1=[r(n)\sqrt{n}]-[r\sqrt{n}]+1}^{[nt\delta]} e^{-Km_1^2/nt} + Cn^{-1/2} \sum_{m_1=[nt\delta]+1}^{\infty} e^{-Km_1} \\ &\quad + Cn^{-1/2} \sum_{m_1=[r(n)\sqrt{n}]+[r\sqrt{n}]+1}^{[nt\delta]} e^{-Km_1^2/nt} + Cn^{-1/2} \sum_{m_1=[nt\delta]+1}^{\infty} e^{-Km_1} \\ &\leq C\int_{r(n)-r}^{\infty} e^{-Kx^2/t}dx + C\int_{r(n)+r}^{\infty} e^{-Kx^2/t}dx + 2\frac{C}{\sqrt{n}}e^{-Knt\delta}.\frac{1}{1-e^{-K}} \to 0 \end{aligned}$$

as $n \to \infty$ (since $r(n) \to \infty$). Therefore, $ES_2^2 \to 0$ as $n \to \infty$. $\square$

Define

$$\Gamma_0^{(2)}((s,q),(t,r)) = \int_{q\vee r}^{\infty} P(\sqrt{\kappa_2}B(s) \le q-x)P(\sqrt{\kappa_2}B(t) \le r-x)dx$$

$$+ \int_{-\infty}^{q\wedge r} P(\sqrt{\kappa_2}B(s) > q-x)P(\sqrt{\kappa_2}B(t) > r-x)dx$$

$$- \mathbf{1}\{r > q\} \int_q^r P(\sqrt{\kappa_2}B(s) \le q-x)P(\sqrt{\kappa_2}B(t) > r-x)dx$$

$$- \mathbf{1}\{q > r\} \int_r^q P(\sqrt{\kappa_2}B(s) > q-x)P(\sqrt{\kappa_2}B(t) \le r-x)dx$$

and

$$\Gamma_q^{(1)}((s,q),(t,r)) = \int_{-\infty}^{\infty} \{P(\sqrt{\kappa_2}B(s) \le q-x)P(\sqrt{\kappa_2}B(t) > r-x) - P(\sqrt{\kappa_2}B(s) \le q-x, \sqrt{\kappa_2}B(t) > r-x)\}dx;$$

$B(\cdot)$ is standard Brownian motion. The covariance terms (2.5) and (2.6) appear in the calculations as $\Gamma_0^{(2)}$ and $\Gamma_q^{(1)}$ resp.

LEMMA 3.2. *S_1 converges to the mean-zero normal distribution with variance*

$$\sigma^2 = \sum_{1\le i,j\le N} \theta_i\theta_j\big\{\rho_0\Gamma_q^{(1)}((t_i,r_i),(t_j,r_j)) + v_0\Gamma_0^{(2)}((t_i,r_i),(t_j,r_j))\big\}.$$

Proof. We apply Lindeberg-Feller to S_1. We check:

1. For any $\epsilon > 0$
$$\lim_{n\to\infty} \sum_{|m|\le[r(n)\sqrt{n}]} E[\bar{U}_m^2\mathbf{1}\{|\bar{U}_m| \ge \epsilon\}] = 0 \tag{3.3}$$
and,

2. $$\lim_{n\to\infty} \sum_{|m|\le[r(n)\sqrt{n}]} E[\bar{U}_m^2] = \sigma^2. \tag{3.4}$$

$|U_m(t,s)| \le n^{-1/4}\eta_0^n(m) + n^{-1/4}\rho_0^n(m)$. By uniform bound on moments (2.2), we get

$$|\bar{U}_m| \le Cn^{-1/4}[\eta_0^n(m) + 1].$$

By choice of $r(n)$ we get (3.3):

$$\lim_{n\to\infty} \sum_{|m|\le[r(n)\sqrt{n}]} E[\bar{U}_m^2 \mathbf{1}\{|\bar{U}_m| \ge \epsilon\}] = 0.$$

We prove (3.4) from the lemma below.

$$\sum_{|m|\le r(n)\sqrt{n}} E\bar{U}_m^2 = \sum_{1\le i,j\le N} \theta_i\theta_j \sum_{|m|\le r(n)\sqrt{n}} E[U_m(t_i,r_i)U_m(t_j,r_j)].$$

Let

$$S = \sum_{|m|\le[r(n)\sqrt{n}]} E[U_m(s,q)U_m(t,r)].$$

LEMMA 3.3.

$$\lim_{n\to\infty} S = \rho_0\Gamma_q^{(1)}((s,q),(t,r)) + v_0\Gamma_0^{(2)}((s,q),(t,r)).$$

Proof.

$$\begin{aligned}
&E[U_m(s,q)U_m(t,r)] \\
&= n^{-1/2}\,\mathrm{Cov}\Bigg[\sum_{j=1}^{\eta_0^n(m)} \left\{\mathbf{1}\{A_{m,j}^{s,q}\}\mathbf{1}\{m > [q\sqrt{n}]\} - \mathbf{1}\{(A_{m,j}^{s,q})^c\}\mathbf{1}\{m \le [q\sqrt{n}]\}\right\}, \\
&\qquad \sum_{j=1}^{\eta_0^n(m)} \left\{\mathbf{1}\{A_{m,j}^{t,r}\}\mathbf{1}\{m > [r\sqrt{n}]\} - \mathbf{1}\{(A_{m,j}^{t,r})^c\}\mathbf{1}\{m \le [r\sqrt{n}]\}\right\}\Bigg].
\end{aligned}$$

Assume z_i are i.i.d. random variables independent of the random nonnegative integer K, then

$$\mathrm{Cov}[\sum_{i=1}^K f(z_i), \sum_{i=1}^K g(z_i)] = EK\cdot \mathrm{Cov}[f(z_1),g(z_1)] + \mathrm{Var}\,K\cdot Ef(z_1)\cdot Eg(z_1).$$

For fixed m, take $K = \eta_0^n(m)$,

$$f(X_{m,j}) = \mathbf{1}\{A_{m,j}^{s,q}\}\mathbf{1}\{m > [q\sqrt{n}]\} - \mathbf{1}\{(A_{m,j}^{s,q})^c\}\mathbf{1}\{m \le [q\sqrt{n}]\}$$

and

$$g(X_{m,j}) = \mathbf{1}\{A_{m,j}^{t,r}\}\mathbf{1}\{m > [r\sqrt{n}]\} - \mathbf{1}\{(A_{m,j}^{t,r})^c\}\mathbf{1}\{m \le [r\sqrt{n}]\}.$$

We now adopt the simpler notation

$$A = \{X(ns) \le [nvs] + [q\sqrt{n}] - m\}$$

and

$$B = \{X(nt) \le [nvt] + [r\sqrt{n}] - m\}.$$

A straightforward computation gives

$$\mathrm{Cov}[f(X_{m,1}), g(X_{m,1})] = P(A)P(B^c) - P(A \cap B^c).$$

Next we compute

$$\begin{aligned} Ef(X_{m,1})Eg(X_{m,1}) =& P(A)P(B)\mathbf{1}\{m > [q\sqrt{n}]\}\mathbf{1}\{m > [r\sqrt{n}\} \\ &+ P(A^c)P(B^c)\mathbf{1}\{m \le [q\sqrt{n}]\}\mathbf{1}\{m \le [r\sqrt{n}\} \\ &- P(A)P(B^c)\mathbf{1}\{m > [q\sqrt{n}]\}\mathbf{1}\{m \le [r\sqrt{n}\} \\ &- P(A^c)P(B)\mathbf{1}\{m \le [q\sqrt{n}]\}\mathbf{1}\{m > [r\sqrt{n}\}. \end{aligned}$$

Putting these computations together we get

$$\begin{aligned} S =& n^{-1/2} \sum_{|m| \le [r(n)\sqrt{n}]} \Bigg\{ \rho_0^n(m)\big\{P(A)P(B^c) - P(A \cap B^c)\big\} \\ &+ v_0^n(m)\Big[P(A)P(B)\mathbf{1}\{m > [q\sqrt{n}] \vee [r\sqrt{n}]\} + P(A^c)P(B^c)\mathbf{1}\{m \le [q\sqrt{n}] \wedge [r\sqrt{n}\} \\ &- P(A)P(B^c)\mathbf{1}\{[q\sqrt{n}] < m \le [r\sqrt{n}\} - P(A^c)P(B)\mathbf{1}\{[r\sqrt{n}] < m \le [q\sqrt{n}]\}\Big]\Bigg\}. \end{aligned}$$

$X(nt)$ is a sum of Poisson(nt) number of independent jumps ξ_i, each jump distributed according to $p(x)$. Therefore,

$$\mathrm{Var}(X(nt)) = nt \cdot \mathrm{Var}(\xi_1) + (E\xi_1)^2 \cdot (nt) = nt(\kappa_2 - v^2) + v^2 nt = nt\kappa_2.$$

By Donsker's Invariance Principle the process $\{(X(nt) - [nvt])/\sqrt{n\kappa_2} : t \ge 0\}$ converges weakly to standard 1-dimensional Brownian motion. Therefore, as $n \to \infty$, assumption (2.3) and a Riemann sum argument together with the large deviation bounds on $X(nt)$

gives us

$$
\begin{aligned}
\lim_{n\to\infty} S = \rho_0 \int_{-\infty}^{\infty} & \{P(B(\kappa_2 s) \leq q-x)P(B(\kappa_2 t) > r-x) \\
& - P(B(\kappa_2 s) \leq q-x, B(\kappa_2 t) > r-x)\}dx \\
+ v_0 \Big\{ & \int_{q\vee r}^{\infty} P(B(\kappa_2 s) \leq q-x)P(B(\kappa_2 t) \leq r-x)dx \\
& + \int_{-\infty}^{q\wedge r} P(B(\kappa_2 s) > q-x)P(B(\kappa_2 t) > r-x)dx \\
& - \int_{q\wedge r}^{q\vee r} \big[\mathbf{1}\{q<r\}P(B(\kappa_2 s) \leq q-x)P(B(\kappa_2 t) > r-x) \\
& + \mathbf{1}\{r<q\}P(B(\kappa_2 s) > q-x)P(B(\kappa_2 t) \leq r-x)\big]\Big\} \\
= \rho_0 \Gamma_q^{(1)} & ((s,q),(t,r)) + v_0 \Gamma_0^{(2)}((s,q),(t,r)).
\end{aligned}
$$

The reasoning behind the convergence of S to the above limit is the same as in Lemmas 4.3 and 4.4 of [13]. The reader is referred to [13] (page 778) for a more detailed explanation.

□

This verifies (3.4):

$$
\lim_{n\to\infty} \sum_{|m|\leq r(n)\sqrt{n}} E[\bar{U}_m^2] = \sum_{1\leq i,j\leq N} \theta_i\theta_j \big\{\rho_0 \Gamma_q^{(1)}((t_i,r_i),(t_j,r_j)) + v_0 \Gamma_0^{(2)}((t_i,r_i),(t_j,r_j))\big\}.
$$

We can now conclude from Lindeberg-Feller that S_1 converges to mean-zero normal distribution with

$$
\sigma^2 = \sum_{1\leq i,j\leq N} \theta_i\theta_j \big\{\rho_0 \Gamma_q^{(1)}((t_i,r_i),(t_j,r_j)) + v_0 \Gamma_0^{(2)}((t_i,r_i),(t_j,r_j))\big\}.
$$

□

LEMMA 3.4.

$$
\lim_{n\to\infty} \sup_{0\leq t\leq T, 0\leq |r|\leq S} n^{-1/4}|EY_n(t,r)| = 0.
$$

Proof. The proof is similar to the proof of Lemma 4.6 (page 781-783) in [13]. □

Showing that

$$
\Gamma_q^{(1)}((s,q),(t,r)) = \Psi_{\kappa_2(t+s)}(|q-r|) - \Psi_{\kappa_2(|s-t|)}(|q-r|) = \Gamma_q((s,q),(t,r))
$$

and

$$\Gamma_0^{(2)}((s,q),(t,r)) = \Gamma_0((s,q),(t,r))$$

is an exercise in calculus.

Proof of Proposition 3.1. By Lemmas 3.1, 3.2 and 3.4 we have, as $n \to \infty$, $n^{-1/4}\sum_{i=1}^N \theta_i Y_n(t_i,r_i)$ converges to a mean-zero normal distribution with variance σ^2, where

$$\sigma^2 = \sum_{1\le i,j\le N} \theta_i\theta_j\{\rho_0\Gamma_q((t_i,r_i),(t_j,r_j)) + v_0\Gamma_0((t_i,r_i),(t_j,r_j))\},$$

for arbitrary $(\theta_1,...,\theta_N) \in \mathbf{R}^N$. We can therefore conclude that as $n \to \infty$ the vector $n^{-1/4}(Y_n(t_1,r_1),Y_n(t_2,r_2),...,Y_n(t_N,r_N))$ converges in distribution to the mean-zero Gaussian random vector $(Z(t_1,r_1),Z(t_2,r_2),...,Z(t_N,r_N))$ with covariance

$$EZ(t_i,r_i)Z(t_j,r_j) = \rho_0\Gamma_q((t_i,r_i),(t_j,r_j)) + v_0\Gamma_0((t_i,r_i),(t_j,r_j)).$$

□

4 Tightness and completion of proof of Theorem 2.1

In this section we first develop a criterion for tightness for processes in D_2. The tightness criterion is in terms of a modulus of continuity. We then proceed to check if our scaled current process satisfies the tightness criterion. The following is an extension of Proposition 5.7 in [6] to two-parameter processes. WLOG for simplicity we replace the region $[0,T]\times[-S,S]$ with the unit square $[0,1]^2$. For any $h \in D_2 = D_2([0,1]^2,\mathbb{R})$, define

$$w_h(\delta) = \sup_{\substack{s,t,q,r\in[0,1]\\ |(s,q)-(t,r)|<\delta}} |h(s,q)-h(t,r)|.$$

PROPOSITION 4.1. *Suppose $\{X_n\}$ is a sequence of random elements of $D_2 = D_2([0,1]^2,\mathbb{R})$ satisfying these conditions:*
For all n there exists $\delta_n > 0$ such that

1. *there exist $\beta > 0$, $\sigma > 2$, and $C > 0$ such that for all n sufficiently large*

$$E(|X_n(s,q)-X_n(t,r)|^\beta) \le C|(s,q)-(t,r)|^\sigma \tag{4.1}$$

for all $s,t,q,r \in [0,1]$ with $|(s,q)-(t,r)| > \delta_n$, and

2. *for every $\epsilon > 0$ and $\eta > 0$ there exists an n_0 such that*

$$P(w_{X_n}(\delta_n) > \epsilon) < \eta \text{ for all } n \ge n_0. \tag{4.2}$$

Then, for each $\epsilon > 0$ and $\eta > 0$, there exists a δ, $0 < \delta < 1$, and an integer n_0, such that

$$P(w_{X_n}(\delta) \geq \epsilon) \leq \eta, \text{ for } n \geq n_0.$$

To prove this proposition we require the following lemma.

LEMMA 4.1. *Let $0 \leq k_0 \leq k$. If points (s, q) and (t, r) lie on the 2^{-k} grid i.e. $s = \frac{i}{2^k}$, $q = \frac{j}{2^k}$, $t = \frac{i'}{2^k}$, $r = \frac{j'}{2^k}$, and $|\frac{i}{2^k} - \frac{i'}{2^k}| \leq 2^{-k_0}$, $|\frac{j}{2^k} - \frac{j'}{2^k}| \leq 2^{-k_0}$, then*

1. *it is possible to move from (s, q) to (t, r) in steps of size 2^{-h}, $k_0 \leq h \leq k$, moving one co-ordinate at a time, where a step of size 2^{-h}, for any h, occurs at most 4 times;*
2. *also, we can choose our steps in such a way that we make a jump of size 2^{-h} only if we lie in the 2^{-h} grid.*

Proof. We fix k_0 and prove the lemma by induction on k. When $k = k_0$, (s, q), (t, r) $\in 2^{-k_0}\mathbb{N} \times 2^{-k_0}\mathbb{N}$. We are given that $|\frac{i}{2^{k_0}} - \frac{i'}{2^{k_0}}| \leq 2^{-k_0}$, $|\frac{j}{2^{k_0}} - \frac{j'}{2^{k_0}}| \leq 2^{-k_0}$. So, either both points coincide or there is a difference of 2^{-k_0} in one co-ordinate or both co-ordinates between these points. We can therefore move from (s, q) to (t, r) in at most 2 steps, each of size 2^{-k_0}, moving one co-ordinate at a time. Clearly condition (2) also holds as the only jumps possible here are of size 2^{-k_0} and we lie in the 2^{-k_0} grid.

Let $k > k_0$, $(s, q) = (\frac{i}{2^k}, \frac{j}{2^k})$, $(t, r) = (\frac{i'}{2^k}, \frac{j'}{2^k})$,$|\frac{i}{2^k} - \frac{i'}{2^k}| \leq 2^{-k_0}$ and $|\frac{j}{2^k} - \frac{j'}{2^k}| \leq 2^{-k_0}$. Either the points (s, q) and (t, r) already lie on the $2^{-(k-1)}$ grid, or if not, the points (s, q) and (t, r) are each at most two jumps of size 2^{-k} away from the $2^{-(k-1)}$-grid. We can therefore move both (s, q) and (t, r) closer to each other and onto the $2^{-(k-1)}$-grid in at most four jumps of size 2^{-k}.

By the induction hypothesis and by our choice of jumps, we are done. □

Proof of Proposition 4.1. We may assume without loss of generality that $\delta_n = 2^{-k}$ for some $k = k(n) \geq 0$ depending on n. This is because if we have $P(w_{X_n}(\delta_n) > \epsilon) < \eta$ for some $\delta_n > 0$, then we can find a $k(n) > 0$ with $2^{-k(n)-1} < \delta_n < 2^{-k(n)}$ such that $w_{X_n}(2^{-k(n)}) \leq 2w_{X_n}(\delta_n)$. So $P(w_{X_n}(2^{-k(n)}) > 2\epsilon) \leq P(w_{X_n}(\delta_n) > \epsilon) < \eta$. Therefore it is sufficient to prove the theorem for $\delta_n = 2^{-k(n)}$ for $k(n) \geq 0$.

Given n and δ, if $\delta \leq \delta_n$, then $w_{X_n}(\delta) \leq w_{X_n}(\delta_n)$. If $\delta_n < \delta$, then for any two points $(s, q), (t, r)$ with $0 \leq |t - s|, |q - r| \leq \delta$, we have

$$|h(s, q) - h(t, r)| \leq |h(s, q) - h(s', q')| + |h(t, r) - h(t', r')| + |h(s', q') - h(t', r')|,$$

where $s', t', q', r' \in \delta_n\mathbb{N}$, $0 \leq |s' - t'|, |q' - r'| \leq \delta$ and $0 \leq |t - t'|, |s - s'|, |r - r'|, |q - q'| \leq \delta_n$. Thus for any $\delta_n < \delta$,

$$w_h(\delta) \leq 2w_h(\delta_n) + \sup_{\substack{t,s,q,r \in (\delta_n\mathbb{N}) \bigcap [0,1] \\ |t-s|,|r-q| \leq \delta}} |h(s, q) - h(t, r)|.$$

Therefore by (4.2), we only need to show that for any $\epsilon > 0$ and $\eta > 0$, there exists a $\delta > 0$ such that for all n sufficiently large

$$P(w^{(n)}_{X_n}(\delta) > \epsilon) < \eta, \tag{4.3}$$

where

$$w^{(n)}_h(\delta) = \sup_{\substack{t,s,q,r \in (\delta_n \mathbb{N}) \bigcap [0,1] \\ |t-s|,|r-q| \leq \delta}} |h(s,q) - h(t,r)|.$$

This follows from (4.1) by the following "dyadic argument":
Choose a λ such that $2^{(2-\sigma)} < \lambda^\beta < 1$. Given n for which (4.1) is satisfied, let

$$\begin{aligned} G_k = \big\{ \big| X_n(\tfrac{i}{2^k}, \tfrac{j}{2^k}) - X_n(\tfrac{i+1}{2^k}, \tfrac{j}{2^k}) \big| \leq \lambda^k \text{ for } i = 0, 1, \ldots, 2^k - 1, \\ j = 0, 1, \ldots, 2^k \big\} \end{aligned}$$

and

$$\begin{aligned} H_k = \big\{ \big| X_n(\tfrac{i}{2^k}, \tfrac{j}{2^k}) - X_n(\tfrac{i}{2^k}, \tfrac{j+1}{2^k}) \big| \leq \lambda^k \text{ for } i = 0, 1, \ldots, 2^k, \\ j = 0, 1, \ldots, 2^k - 1 \big\} \end{aligned}$$

where $k \leq k(n)$.

$$P(G_k^c \cup H_k^c) \leq 2^k (2^k + 1) \lambda^{-k\beta} 2^{-k\sigma} + 2^k (2^k + 1) \lambda^{-k\beta} 2^{-k\sigma}$$

by Markov inequality and (4.1) as $k \leq k(n)$

$$\begin{aligned} &\leq c (2^{(2-\sigma)} \lambda^{-\beta})^k \\ &= c\gamma^k \text{ where } \gamma = 2^{(2-\sigma)} \lambda^{-\beta} < 1. \end{aligned}$$

Given $\epsilon > 0$ and $\eta > 0$, choose k_0 such that

$$c \sum_{k \geq k_0} \gamma^k < \eta \text{ and } 4 \sum_{k \geq k_0} \lambda^k < \epsilon.$$

Choose $\delta = 2^{-k_0}$. If $\delta < \delta_n$ for some $n \geq n_0$, then $w_{X_n}(\delta) < w_{X_n}(\delta_n)$ and we have $P(w_{X_n}(\delta) > \epsilon) < \eta$ by (4.2). A little more work is required to show that (4.3) holds in the $\delta_n \leq \delta$, i.e. $k_0 \leq k(n)$, case. Pick any $(s,q), (t,r) \in \delta_n \mathbb{N} \times \delta_n \mathbb{N}$ where $\delta_n = 2^{-k(n)}$, such that $|s-t|, |q-r| < \delta$. We can find a sequence of points $(s,q) = (s_1, q_1), (s_2, q_2) ... (s_m, q_m) = (t,r)$ (refer to lemma 4.1) such that on the event $\bigcap_{k_0 \leq k \leq k(n)} (G_k \cap H_k)$ we have

$$|X_n(s,q) - X_n(t,r)| \leq \sum_{i=1}^{m-1} |X_n(s_i, q_i) - X_n(s_{i+1}, q_{i+1})| \leq 4 \sum_{k \geq k_0} \lambda^k < \epsilon.$$

Now

$$P\Big(\bigcup_{k_0\le k\le k(n)}(G_k^c\cup H_k^c)\Big)\le c\sum_{k_0\le k\le k(n)}\gamma^k<\eta.$$

Therefore,

$$P(w_{X_n}^{(n)}(\delta)\le\epsilon)\ge P\Big(\bigcap_{k_0\le k\le k(n)}(G_k\cap H_k)\Big)\ge 1-\eta.$$

Thus (4.3) is satisfied with $\delta=2^{-k_0}$. □

We now apply Proposition 4.1 to processes $X_n=n^{-1/4}\bar{Y}_n(t,r)$ where $\bar{Y}_n(t,r)=Y_n(t,r)-EY_n(t,r)$.

4.1 Verifying the first tightness condition

We check that (4.1) holds for $n^{-1/4}\bar{Y}_n(t,r)$. Let $\alpha>0$ and

$$5/4+\alpha<\beta<3/2. \tag{4.4}$$

We show that there exist constants $\sigma>2$ and $0<C<\infty$ independent of n, such that with $\delta_n=n^{-\beta}$, for all n (sufficiently large)

$$E\left(n^{-1/4}|\bar{Y}_n(s,q)-\bar{Y}_n(t,r)|\right)^{12}\le C|(s,q)-(t,r)|^{\sigma} \tag{4.5}$$

for all $(s,q),(t,r)\in[0,T]\times[-S,S]$ with $|(s,q)-(t,r)|>\delta_n$.

We can assume WLOG that $s\le t$ in the following calculations. Define

$$A_{m,j}=\left\{X_{m,j}(ns)\le[q\sqrt{n}\,]+[nvs],X_{m,j}(nt)>[r\sqrt{n}\,]+[nvt]\right\}$$

and

$$B_{m,j}=\left\{X_{m,j}(ns)>[q\sqrt{n}\,]+[nvs],X_{m,j}(nt)\le[r\sqrt{n}\,]+[nvt]\right\}.$$

If $q \le r$ then

$$Y_n(s,q) - Y_n(t,r) = \sum_{m>[r\sqrt{n}]} \sum_{j=1}^{\eta_0^n(m)} \mathbf{1}\{X_{m,j}(ns) \le [q\sqrt{n}] + [nvs]\} \tag{4.6}$$

$$+ \sum_{m=[q\sqrt{n}]+1}^{[r\sqrt{n}]} \sum_{j=1}^{\eta_0^n(m)} \mathbf{1}\{X_{m,j}(ns) \le [q\sqrt{n}] + [nvs]\} \tag{4.7}$$

$$- \sum_{m\le[q\sqrt{n}]} \sum_{j=1}^{\eta_0^n(m)} \mathbf{1}\{X_{m,j}(ns) > [q\sqrt{n}] + [nvs]\} \tag{4.8}$$

$$- \sum_{m>[r\sqrt{n}]} \sum_{j=1}^{\eta_0^n(m)} \mathbf{1}\{X_{m,j}(nt) \le [r\sqrt{n}] + [nvt]\} \tag{4.9}$$

$$+ \sum_{m=[q\sqrt{n}]+1}^{[r\sqrt{n}]} \sum_{j=1}^{\eta_0^n(m)} \mathbf{1}\{X_{m,j}(nt) > [r\sqrt{n}] + [nvt]\} \tag{4.10}$$

$$+ \sum_{m\le[q\sqrt{n}]} \sum_{j=1}^{\eta_0^n(m)} \mathbf{1}\{X_{m,j}(nt) > [r\sqrt{n}] + [nvt]\} \tag{4.11}$$

Combining (4.6) and (4.9), (4.7) and (4.10), (4.8) and (4.11) and adding and subtracting

$$\sum_{m=[q\sqrt{n}]+1}^{[r\sqrt{n}]} \sum_{j=1}^{\eta_0^n(m)} \mathbf{1}\{X_{m,j}(ns) > [q\sqrt{n}] + [nvs], X_{m,j}(nt) \le [r\sqrt{n}] + [nvt]\}$$

we get

$$\bar{Y}_n(s,q) - \bar{Y}_n(t,r) = \sum_{m\in\mathbb{Z}} G_m + \sum_{m=[q\sqrt{n}\ ]+1}^{[r\sqrt{n}\ ]} \big[\eta_0^n(m) - \rho_0^n(m)\big]$$

where

$$G_m = \sum_{j=0}^{\eta_0^n(m)} (\mathbf{1}_{A_{m,j}} - \mathbf{1}_{B_{m,j}}) - \rho_0^n(m)(P(A_{m,1}) - P(B_{m,1})). \tag{4.12}$$

Similarly, when $q > r$

$$\bar{Y}_n(s,q) - \bar{Y}_n(t,r) = \sum_{m\in\mathbb{Z}} G_m - \sum_{m=[r\sqrt{n}\ ]+1}^{[q\sqrt{n}\ ]} \big[\eta_0^n(m) - \rho_0^n(m)\big].$$

Using the identity $(a+b)^k \leq 2^k(a^k + b^k)$, we get

$$E\left(n^{-1/4}(\bar{Y}_n(s,q) - \bar{Y}_n(t,r))\right)^{12} \leq \frac{2^{12}}{n^3}\left(EA^{12} + EB^{12}\right) \tag{4.13}$$

where

$$A = \sum_{m\in\mathbb{Z}} G_m$$

and

$$B = \sum_{m=[(r\wedge q)\sqrt{n}\ ]+1}^{[(r\vee q)\sqrt{n}\ ]} \left[\eta_0^n(m) - \rho_0^n(m)\right].$$

We now bound $E[A^{12}]$ and $E[B^{12}]$.

To bound EB^{12}, we use the following lemma which is a slight modification of Lemma 8 in [10].

LEMMA 4.2. *Let Y_i be independent random variables with $E[|Y_i|^{2r}] < c < \infty$ and $EY_i = 0$ for all i and for some fixed $r > 0$. There is a constant $C < \infty$ such that, for any n ,*

$$E\left\{\left(Y_1 + Y_2 + \cdots + Y_n\right)^{2r}\right\} \leq C(2r)!n^r.$$

Proof. Since $EY_i = 0$,

$$E\left\{\left(Y_1 + Y_2 + \cdots + Y_n\right)^{2r}\right\} = \sum{}' \frac{(2r)!}{r_1!r_2!\cdots r_n!} EY_1^{r_1}EY_2^{r_2}\cdots EY_n^{r_n}$$

where $\sum'$ extends over all n-tuples of integers $r_1, r_2, \ldots r_n \geq 0$ such that each $r_i \neq 1$ and $r_1 + \cdots + r_n = 2r$.
$|EY_1^{r_1}EY_2^{r_2}\cdots EY_n^{r_n}| \leq c$ by the bounded moment assumption and Hölder's inequality. The number $A = \sum'(r_1!\cdots r_n!)^{-1}$ is the coefficient of x^{2r} in

$$F_n(x) = \Big(\sum_{\substack{j\geq 0\\ j\neq 1}} \frac{x^j}{j!}\Big)^n$$

and consequently $A \leq \frac{F_n(x)}{x^{2r}}$ for $x > 0$. But $\sum_{\substack{j\geq 0\\ j\neq 1}} \frac{x^j}{j!} \leq 1 + x^2$ for $x \leq 1$, so taking $x = n^{-1/2}$ gives $A \leq n^r(1 + (1/n))^n \leq Cn^r$. □

Recall that $\eta_0^n(x), x \in \mathbb{Z}$ are independent with mean $\rho_0^n(x)$. Let C denote a constant that varies from line to line in the string of inequalities below. Applying Lemma 4.2 to the term $E[B^{12}]$ in equation (4.13) with $r = 6$ and by the moment assumption on η_0^n, we get

$$E[B^{12}] \leq C(12)!|[r\sqrt{n}] - [q\sqrt{n}]|^6 \leq C(|r-q|^6 n^3 + 1). \tag{4.14}$$

We use the following lemma to bound $E[A^{12}]$.

LEMMA 4.3. *There exists a constant C such that for each positive integer $1 \le k \le 12$ and for all m,*

$$E[|G_m|^k] \le D_m$$

where $D_m = C\{P(A_{m,1}) + P(B_{m,1})\}$.

Proof. The proof is the same as in Lemma 4.7 (pages 784-785) of [13]. □

LEMMA 4.4. *There exists a $c > 0$ such that*

$$E[A^{12}] \le c \left\{ 1 + \left(\sum_m D_m \right)^6 \right\}. \tag{4.15}$$

Proof. $A = \sum_{m\in\mathbb{Z}} G_m$. Note that $EG_m = 0$ for all m and the G_m's are independent. Let $\sum^{(k)}$ denote the sum over all k-tuples of integers $r_1, r_2, \ldots, r_k \ge 2$ such that $r_1 + \cdots + r_k = 12$.

$$\begin{aligned} E[A^{12}] &= \sum_{m_1,\ldots,m_{12}\in\mathbb{Z}} E[G_{m_1} G_{m_2} \cdots G_{m_{12}}] \\ &\le \sum_{k=1}^{6} \sum\nolimits^{(k)} \frac{12!}{r_1! \cdots r_k!} \sum_{m_1 \ne m_2 \ne \cdots m_k} E|G_{m_1}|^{r_1} E|G_{m_2}|^{r_2} \cdots E|G_{m_k}|^{r_k} \qquad (4.16) \\ &\le \sum_{k=1}^{6} \sum\nolimits^{(k)} \frac{12!}{r_1! r_2! \cdots r_k!} \sum_m E|G_m|^{r_1} \sum_m E|G_m|^{r_2} \cdots \sum_m E|G_m|^{r_k}. \end{aligned}$$

Since $E|G_m|^l \le D_m$ for all $m \in \mathbb{Z}$ and $1 \le l \le 12$ we get, for all $1 \le k \le 6$,

$$\sum_m E|G_m|^{r_1} \sum_m E|G_m|^{r_2} \cdots \sum_m E|G_m|^{r_k} \le \max\{1, (\sum_m D_m)^6\} \le \{1 + (\sum_m D_m)^6\}.$$

Thus

$$E[A^{12}] \le c \left\{ 1 + \left(\sum_m D_m \right)^6 \right\}. \tag{4.17}$$

□

We evaluate $\sum_{m\in\mathbb{Z}} D_m$ below. Recall that

$$\sum_{m\in\mathbb{Z}} D_m = C \left(\sum_{m\in\mathbb{Z}} P(A_{m,1}) + \sum_{m\in\mathbb{Z}} P(B_{m,1}) \right).$$

$$
\begin{aligned}
\sum_{m\in\mathbb{Z}} P(A_{m,1}) &= \sum_{m\in\mathbb{Z}} P(X(ns) \le [q\sqrt{n}] + [nvs] - m, X(nt) > [r\sqrt{n}] + [nvt] - m) \\
&= \sum_{m\in\mathbb{Z}} \sum_{l\ge m} \{P(X(ns) = [q\sqrt{n}] + [nvs] - l) \\
&\qquad \times P(X(nt) - X(ns) > [nvt] - [nvs] + [r\sqrt{n}] - [q\sqrt{n}] + l - m)\} \\
&= \sum_{l\in\mathbb{Z}} \sum_{k\ge 0} \{P(X(ns) = [q\sqrt{n}] + [nvs] - l) \\
&\qquad \times P(X(n(t-s)) > [nvt] - [nvs] + [r\sqrt{n}] - [q\sqrt{n}] + k)\} \\
&\le \sum_{k\ge 0} P(X(n(t-s)) - [nv(t-s)] > [r\sqrt{n}] - [q\sqrt{n}] + k)
\end{aligned}
\tag{4.18}
$$

Similarly,

$$
\sum_{m\in\mathbb{Z}} P(B_{m,1}) \le \sum_{k<0} P(X(n(t-s)) - [nv(t-s)] \le [r\sqrt{n}] - [q\sqrt{n}] + k + 1) \tag{4.19}
$$

Together,

$$
\begin{aligned}
\sum_{m\in\mathbb{Z}} P(A_{m,1}) + \sum_{m\in\mathbb{Z}} P(B_{m,1}) &\le E\big|X(n(t-s)) - [nv(t-s)] + [q\sqrt{n}] - [r\sqrt{n}]\big| \\
&\le E\big|X(n(t-s)) - [nv(t-s)]\big| + |r-q|\sqrt{n} + 1 \\
&\le c\{\sqrt{n(t-s)} + |r-q|\sqrt{n} + 1\}.
\end{aligned}
$$

Consequently, (4.15) becomes

$$
E[A^{12}] \le c\{1 + (n(t-s))^3 + |r-q|^6 n^3\}. \tag{4.20}
$$

Putting (4.14) and (4.20) together in (4.13) we get

$$
E[\{n^{-1/4}(\bar{Y}_n(s,q) - \bar{Y}_n(t,r))\}^{12}] \le c\big(n^{-3} + (t-s)^3 + |r-q|^6\big)
$$

Using $\beta < 3/2$, $|r-q| \le 2S < \infty$ and $t - s \le T < \infty$, we can find constants $c > 0$ and $\sigma > 2$ such that

$$
E[\{n^{-1/4}(\bar{Y}_n(s,q) - \bar{Y}_n(t,r))\}^{12}\ ] \le c(\ |r-q|^\sigma + |t-s|^\sigma),
$$

if $|r-q| > n^{-\beta}$ or $t - s > n^{-\beta}$. This verifies the first tightness condition (4.1).

4.2 Verifying the second tightness condition

To verify (4.2) for $n^{-1/4}\bar{Y}_n(t,r)$, it is sufficient to show

LEMMA 4.5. *For any $0 < T, S < \infty$ and $\epsilon > 0$,*

$$\lim_{n\to\infty} P\Bigg\{ \bigcup_{\substack{0\le k_1\le [Tn^\beta] \\ [-n^\beta S]\le k_2\le [n^\beta S]}} \Bigg[\sup_{\substack{k_1 n^{-\beta}\le t\le (k_1+1)n^{-\beta} \\ k_2 n^{-\beta}\le r\le (k_2+1)n^{-\beta}}} |Y_n(t,r) - Y_n(n^{-\beta}k_1, n^{-\beta}k_2)| \ge n^{1/4}\epsilon \Bigg] \Bigg\} = 0. \tag{4.21}$$

Proof. Recall from (4.4) that $5/4 + \alpha < \beta < 3/2$, $\alpha > 0$. We first show that particles starting at a distance of $n^{1/2+\alpha}$ or more from the interval $([-(S+1)\sqrt{n}], [S\sqrt{n}])$ do not contribute to $Y_n(\cdot, r)$ during time interval $[0,T]$, $r \in [-S,S]$, in the $n \to \infty$ limit.

LEMMA 4.6. *Let*

$$\begin{aligned} N_1 = &\sum_{m\le [-(S+1)\sqrt{n}]-n^{1/2+\alpha}} \sum_{j=1}^{\eta_0^n(m)} \mathbf{1}\{X_{m,j}(nt) \ge -[(S+1)\sqrt{n}] + [nvt] \\ &\qquad\qquad \textit{for some } 0\le t\le T\} \\ &+ \sum_{m\ge [S\sqrt{n}]+n^{1/2+\alpha}} \sum_{j=1}^{\eta_0^n(m)} \mathbf{1}\{X_{m,j}(nt) \le [S\sqrt{n}] + [nvt] \\ &\qquad\qquad \textit{for some } 0\le t\le T\}. \end{aligned} \tag{4.22}$$

Then $EN_1 \to 0$ as $n \to \infty$.

Proof. Choose a positive integer M large enough so that $1/2 - \alpha(2M-1) < 0$. The expectation of the first sum in (4.22) is bounded by

$$\begin{aligned} &C \sum_{m\le [-(S+1)\sqrt{n}]-n^{1/2+\alpha}} P(\sup_{0\le t\le T} (X(nt) - nvt) \ge -[(S+1)\sqrt{n}] - 1 - m) \\ &\le C \sum_{l\ge n^{1/2+\alpha}} P(\sup_{0\le t\le T} (X(nt) - nvt) \ge l-1) \\ &\le C \sum_{l\ge n^{1/2+\alpha}} l^{-2M} E[(X(nT) - nvT)_+^{2M}] \end{aligned}$$

by application of Doob's inequality to the martingale $X(t) - vt$

$$\le C \sum_{l\ge n^{1/2+\alpha}} l^{-2M} n^M$$

as $E[(X(nT) - nvT)_+^{2M}]$ is $\mathrm{O}(n^M)$

$$\leq Cn^{1/2-\alpha(2M-1)} \to 0 \text{ as } n \to \infty.$$

Similarly the expectation of the other sum goes to 0 as $n \to \infty$. □

Let

$$N_2 = \sum_{\substack{m=[-(S+1)\sqrt{n}]-n^{1/2+\alpha};\\ m\in\mathbb{Z}}}^{([S\sqrt{n}])+n^{1/2+\alpha}} \eta_0^n(m).$$

be the number of particles initially within $n^{1/2+\alpha}$ distance of the interval $(-[(S+1)\sqrt{n}], [S\sqrt{n}])$. Fix a constant c so that

$$\lim_{n\to\infty} P(N_2 \geq cn^{1/2+\alpha}) = 0. \tag{4.23}$$

Consider the event

$$\bigcup_{\substack{0\leq k_1\leq [Tn^\beta]\\ [-n^\beta S]\leq k_2\leq [n^\beta S]}} \left\{ \sup_{\substack{k_1 n^{-\beta}\leq t\leq (k_1+1)n^{-\beta}\\ k_2 n^{-\beta}\leq r\leq (k_2+1)n^{-\beta}}} |Y_n(t,r) - Y_n(n^{-\beta}k_1, n^{-\beta}k_2)| \geq n^{1/4}\epsilon \right\}. \tag{4.24}$$

If $t_0 = n^{-\beta}k_1$ and $r_0 = n^{-\beta}k_2$, then

$$|Y_n(t,r) - Y_n(t_0,r_0)| \leq |Y_n(t,r) - Y_n(t_0,r)| + |Y_n(t_0,r) - Y_n(t_0,r_0)|.$$

For fixed k_1 and k_2, the event in braces in (4.24) is contained in the following union of two events:

$$\left\{ \sup_{t_0\leq t\leq t_0+n^{-\beta}} \sup_{r_0\leq r\leq r_0+n^{-\beta}} |Y_n(t,r) - Y_n(t_0,r)| \geq \frac{1}{2}\epsilon n^{1/4} \right\} \tag{4.25a}$$

$$\bigcup \left\{ \sup_{r_0\leq r\leq r_0+n^{-\beta}} |Y_n(t_0,r) - Y_n(t_0,r_0)| \geq \frac{1}{2}\epsilon n^{1/4} \right\} \tag{4.25b}$$

The first event (4.25a) implies that at least one of the following two things happen:

1. At least $\frac{1}{4}\epsilon n^{1/4}$ particles cross the discretized characteristic

$$s \mapsto [r\sqrt{n}] + [nvs]$$

for some $r \in [r_0, r_0+n^{-\beta}]$, during time interval $s \in [t_0, (t_0+n^{-\beta})]$ by jumping. On the event $\{N_1 = 0\}$, these particles must be among the N_2 particles initially within $n^{1/2+\alpha}$

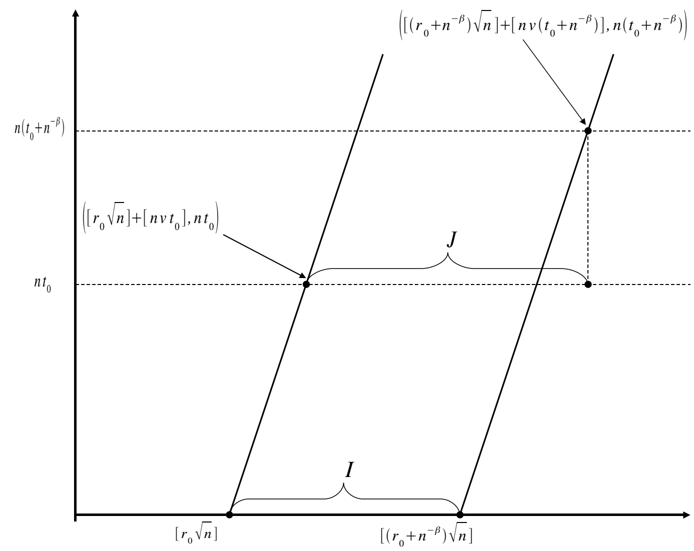

Figure 1: Two characteristic lines at distance $\delta_n\sqrt{n} = n^{-\beta}\sqrt{n}$ apart.

distance of the interval $(-[(S+1)\sqrt{n}], [S\sqrt{n}])$. Therefore, conditioned on $\{N_1 = 0\}$, the probability of this event is bounded by the probability that N_2 independent rate 1 random walks altogether experience at least $\frac{1}{4}\epsilon n^{1/4}$ jumps in time interval of length $n^{1-\beta}$.

2. At least $\frac{1}{4}\epsilon n^{1/4}$ particles cross the discretized characteristic

$$s \mapsto [r\sqrt{n}] + [nvs]$$

for some $r \in [r_0, r_0 + n^{-\beta}]$, during time interval $[t_0, (t_0 + n^{-\beta})]$ by staying put while the characteristic crosses the location of these particles. These particles must lie in the interval J at time nt_0, where

$$J = \Big[[r_0\sqrt{n}\ ] + [nvt_0], [(r_0 + n^{-\beta})\sqrt{n}\ ] + [nv(t_0 + n^{-\beta})]\Big],$$

(refer to figure 1). For large enough n, the distance between the endpoints of J is at most 2. So these $\frac{1}{4}\epsilon n^{1/4}$ particles must sit on at most 2 sites, say $x^1_{k_1,k_2}$ and $x^2_{k_1,k_2}$.

The second event (4.25b) implies that at least $\frac{1}{2}\epsilon n^{1/4}$ particles either lie in the interval $\big[[r_0\sqrt{n}] + [nvt_0], [r\sqrt{n}] + [nvt_0]\big]$ at time nt_0, or lie in the interval $\big[[r_0\sqrt{n}], [r\sqrt{n}]\big]$ at time 0. Since

$$\big[[r_0\sqrt{n}] + [nvt_0], [r\sqrt{n}] + [nvt_0]\big] \subseteq J$$

and

$$\big[[r_0\sqrt{n}], [r\sqrt{n}]\big] \subseteq I$$

where $I = \big[[r_0\sqrt{n}\ ], [(r_0 + n^{-\beta})\sqrt{n}\ ]\big]$ (refer to figure 1), this event implies that at least $\frac{1}{4}\epsilon n^{1/4}$ particles lie in interval J at time nt_0 or at least $\frac{1}{4}\epsilon n^{1/4}$ particles lie in interval I at time 0. For large enough n, the distance between the endpoints of I is at most 1, so the $\frac{1}{4}\epsilon n^{1/4}$ particles lying in interval I at time 0 must sit on a unique site x^0_{0,k_2} say.

Let $\Pi(cn^{3/2+\alpha-\beta})$ denote a mean $cn^{3/2+\alpha-\beta}$ Poisson random variable that represents the total number of jumps among $cn^{1/2+\alpha}$ independent particles during a time interval of length $n^{1-\beta}$. Then,

$$\begin{aligned}
&P\,(\text{ event in } (4.21)) \\
&\leq P\,(N_1 \geq 1) + P\left(N_2 \geq cn^{1/2+\alpha}\right) \\
&+ \sum_{k_1=0}^{[Tn^\beta]} \sum_{k_2=[-Sn^\beta]}^{[Sn^\beta]} \Bigg\{ P\left(\Pi(cn^{3/2+\alpha-\beta}) \geq \tfrac{1}{4}\epsilon n^{1/4}\right) + P\left(\eta^n_0(x^0_{0,k_2}) \geq \tfrac{1}{4}\epsilon n^{1/4}\right) \\
&+ 2\big\{ P\left(\eta^n_{n^{1-\beta}k_1}(x^1_{k_1,k_2}) \geq \tfrac{1}{8}\epsilon n^{1/4}\right) + P\left(\eta^n_{n^{1-\beta}k_1}(x^2_{k_1,k_2}) \geq \tfrac{1}{8}\epsilon n^{1/4}\right)\big\}\Bigg\}.
\end{aligned} \tag{4.26}$$

The probabilities $P(N_1 \geq 1)$ and $P(N_2 \geq cn^{1/2+\alpha})$ vanish as $n \to \infty$ by lemma 4.6 and (4.23). $\Pi(cn^{3/2+\alpha-\beta})$ is stochastically bounded by a sum of $M_n = [cn^{3/2+\alpha-\beta}] + 1$ i.i.d. mean 1 Poisson variables, and so a standard large deviation estimate gives

$$P\left(\Pi(cn^{3/2+\alpha-\beta}) \geq \tfrac{1}{4}\epsilon n^{1/4}\right) \leq \exp\left\{-M_n I(\tfrac{1}{4}M_n^{-1}\epsilon n^{1/4})\right\},$$

where I is the Cramér rate function for the Poisson(1) distribution. By the choice of α and β, $M_n \geq n^\alpha$, while $M_n^{-1}n^{1/4} \to \infty$. Consequently, there are constants $0 < C_0, C_1 < \infty$ such that

$$\sum_{k_1=0}^{[Tn^\beta]} \sum_{k_2=[-Sn^\beta]}^{[Sn^\beta]} P\left(\Pi(cn^{3/2+\alpha-\beta}) \geq \tfrac{1}{4}\epsilon n^{1/4}\right) \leq C_0 n^{2\beta} \exp\{-C_1 n^\alpha\} \to 0.$$

By Lemma 4.10 in [13], we have

$$\sup_{x\in\mathbb{Z}, t\geq 0} E[\eta_t(x)^{12}] < \infty. \tag{4.27}$$

So,

$$\begin{aligned}
&\sum_{k_1=0}^{[Tn^\beta]} \sum_{k_2=[-Sn^\beta]}^{[Sn^\beta]} P(\eta^n_{k_1 n^{1-\beta}}(x^i_{k_1,k_2}) \geq \tfrac{1}{8}\epsilon n^{1/4}) \\
&\qquad\qquad \leq (Tn^\beta + 1)(2Sn^\beta + 1) 8^{12} \epsilon^{-12} n^{-3} \sup_{x,t,n} E[\eta^n_t(x)^{12}]
\end{aligned}$$

vanishes as $n \to \infty$ by (4.27) and because $2\beta - 3 < 0$.
Similarly for the other probability in (4.26).

□

Since the two conditions of Theorem 4.1 hold for the sequence of processes $\{n^{-1/4}\bar{Y}_n\}$, we can conclude that

$$\lim_{\delta \downarrow 0} \limsup_n P\{w_{n^{-1/4}\bar{Y}_n}(\delta) \geq \epsilon\} = 0 \text{ for all } \epsilon > 0. \tag{4.28}$$

4.3 Weak Convergence

Finally, we use the theorem about weak convergence in D_2 from [2]. By Theorem 2 in [2] we have X_n converges weakly to X in D_2 if,

1. $(X_n(t_1, r_1), \cdots, X_n(t_N, r_N))$ converges weakly to $(X(t_1, r_1), \cdots, X(t_N, r_N))$ for all finite subsets $\{(t_i, r_i)\} \in [0, T] \times [-S, S]$, and
2. $\lim_{\delta \to 0} \limsup_n P\{w_{X_n}(\delta) \geq \epsilon\} = 0$ for all $\epsilon > 0$, where

$$w_x(\delta) = \sup_{\substack{(s,q),(t,r) \in [0,T] \times [-S,S] \\ |(s,q)-(t,r)| < \delta}} |x(s, q) - x(t, r)|.$$

This, together with the convergence of finite-dimensional distributions of $\{n^{-1/4}\bar{Y}_n(\cdot, \cdot)\}$ and (4.28) gives us weak convergence of $\{n^{-1/4}\bar{Y}_n(\cdot, \cdot)\}$ as $n \to \infty$. Since the expectations $n^{-1/4}EY_n(t, s)$ vanish uniformly over $0 \leq t \leq T$, $0 \leq |s| \leq S$ by (3.4), we conclude that the process $\{n^{-1/4}Y_n(\cdot, \cdot)\}$ converges weakly as $n \to \infty$.

5 Proof of large deviation results

Proof of Theorem 2.2 and Corollary 2.2. Assume that $\eta_0^n(m), m \in \mathbb{Z}$ are i.i.d. Fix $r \in \mathbb{R}$ and $t > 0$.We prove that $n^{-1/2}Y_n(t, r)$ satisfies the LDP with a good rate function. We start with some preliminary calculations.

$$\begin{aligned} Y_n(t, r) = & \sum_{m=-\infty}^{\infty} \sum_{j=1}^{\eta_0^n(m)} \big[\mathbf{1}\{X_{m,j}(nt) \leq [nvt] + [r\sqrt{n}]\}\mathbf{1}\{m > [r\sqrt{n}]\} \\ & - \mathbf{1}\{X_{m,j}(nt) > [nvt] + [r\sqrt{n}]\}\mathbf{1}\{m \leq [r\sqrt{n}]\}\big] \\ = & \sum_{m=-\infty}^{\infty} \sum_{j=1}^{\eta_0^n(m)} [f_{m,j}^{n,(1)}(t, r) - f_{m,j}^{n,(2)}(t, r)]. \end{aligned} \tag{5.1}$$

Define

$$M_m^n(\lambda) = Ee^{\lambda[f_{m,1}^{n,(1)}(t,r) - f_{m,1}^{n,(2)}(t,r)]}. \tag{5.2}$$

If $m > [r\sqrt{n}]$ then

$$\begin{aligned} M_m^n(\lambda) &= Ee^{\lambda f_{m,1}^{n,(1)}(t,r)} = E\left[\sum_{k\geq 0} \frac{(\lambda f_{m,1}^{n,(1)}(t,r))^k}{k!}\right] \\ &= 1 + (e^{\lambda} - 1)E[f_{m,1}^{n,(1)}(t,r)] \\ &= 1 + (e^{\lambda} - 1)P(X(nt) \leq [nvt] + [r\sqrt{n}] - m) \end{aligned} \tag{5.3}$$

where $X(\cdot)$ represents a random walk with rates $p(x)$ starting at the origin. Similarly, if $m \leq [r\sqrt{n}]$ then

$$\begin{aligned} M_m^n(\lambda) &= Ee^{-\lambda f_{m,1}^{n,(2)}(t,r)} = E\left[\sum_{k\geq 0} \frac{(-\lambda \cdot f_{m,1}^{n,(2)}(t,r))^k}{k!}\right] \\ &= 1 + (e^{-\lambda} - 1)P(X(nt) > [nvt] + [r\sqrt{n}] - m). \end{aligned} \tag{5.4}$$

We now calculate the logarithmic moment generating function for $Y_n(t,r)$.

$$\begin{aligned} \log Ee^{\lambda Y_n(t,r)} = &\sum_{|m-[r\sqrt{n}]|\leq[nt\delta]} \log E\exp\left\{\lambda \sum_{j=1}^{\eta_0^n(m)} [f_{m,j}^{n,(1)}(t,r) - f_{m,j}^{n,(2)}(t,r)]\right\} \\ &+ \sum_{|m-[r\sqrt{n}]|>[nt\delta]} \log E\exp\left\{\lambda \sum_{j=1}^{\eta_0^n(m)} [f_{m,j}^{n,(1)}(t,r) - f_{m,j}^{n,(2)}(t,r)]\right\} \end{aligned}$$

(By large deviation bounds on $X(nt)$ (3.1) and (3.2) , the second term is of $o(\sqrt{n})$)

$$= \sum_{|m-[r\sqrt{n}]|\leq[nt\delta]} \log E\exp\left\{\lambda \sum_{j=1}^{\eta_0^n(m)} [f_{m,j}^{n,(1)}(t,r) - f_{m,j}^{n,(2)}(t,r)]\right\} + o(\sqrt{n})$$

$$
\begin{aligned}
\lim_{n\to\infty}\frac{1}{\sqrt{n}}\log Ee^{\lambda Y_n(t,r)} &= \lim_{n\to\infty}\frac{1}{\sqrt{n}}\sum_{|m-r\sqrt{n}|\le[nt\delta]}\log E\left[e^{\lambda\sum_{j=1}^{\eta_0^n(m)}[f_{m,j}^{n,(1)}(t,r)-f_{m,j}^{n,(2)}(t,r)]}\right]\\
&= \lim_{n\to\infty}\frac{1}{\sqrt{n}}\sum_{|m-r\sqrt{n}|\le[nt\delta]}\log\sum_{k\ge 0}P(\eta_0^n(m)=k)(M_m^n(\lambda))^k
\end{aligned}
$$

(Recall that $\gamma(\alpha)=\log Ee^{\alpha\eta_0^n(\cdot)}$)

$$
= \lim_{n\to\infty}\frac{1}{\sqrt{n}}\sum_{|m-r\sqrt{n}|\le[nt\delta]}\gamma(\log M_m^n(\lambda)). \tag{5.5}
$$

Using (5.3), (5.4), (5.5), applying Central Limit Theorem and a Riemann sum argument we get,

$$
\begin{aligned}
\Lambda(\lambda) &= \lim_{n\to\infty}\frac{1}{\sqrt{n}}\log Ee^{\lambda Y_n(t,r)}\\
&= \int_0^\infty \gamma(\log\{1+(e^\lambda-1)\Phi_{\kappa_2 t}(-x)\})dx + \int_{-\infty}^0 \gamma(\log\{1+(e^{-\lambda}-1)\Phi_{\kappa_2 t}(x)\})dx.
\end{aligned} \tag{5.6}
$$

By Assumption 2.3 we get $\Lambda(\lambda)<\infty$ for $\lambda\in\mathbb{R}$. It is also easy to check that $\Lambda(\lambda)$ is strictly convex and essentially smooth on $\mathbb{R}$. By Theorem 2.3.6 in [5] (Gärtner-Ellis theorem) $I(\cdot)$, the convex dual of $\Lambda(\lambda)$, is the good rate function. We now find the explicit expression for the rate function.

If $\eta_0^n(m)\sim$ Poisson(ρ), then $\gamma(\alpha)=\rho(e^\alpha-1)$. Therefore,

$$
\begin{aligned}
\Lambda(\lambda) &= \rho(e^\lambda-1)\int_0^\infty \Phi_{\kappa_2 t}(-x)dx + \rho(e^{-\lambda}-1)\int_{-\infty}^0 \Phi_{\kappa_2 t}(x)dx\\
&= \rho\sqrt{\frac{\kappa_2 t}{2\pi}}(e^\lambda+e^{-\lambda}-2).
\end{aligned}
$$

The convex dual of this is:

$$
\begin{aligned}
I(x) &= \sup_{\lambda\in\mathbb{R}}\{x\lambda-\Lambda(\lambda)\}\\
&= x\log\left(\frac{x\sqrt{\pi}}{\rho\sqrt{2\kappa_2 t}}+\sqrt{1+\frac{x^2\pi}{2\rho^2\kappa_2 t}}\right)-\rho\sqrt{\frac{2\kappa_2 t}{\pi}}\left(\sqrt{1+\frac{x^2\pi}{2\rho^2\kappa_2 t}}-1\right) \text{ for } x\in\mathbb{R}.
\end{aligned}
$$

This proves (2.15).

To prove Theorem 2.2 we first check that $I(\cdot)$ given by (2.14) is the convex dual of $\Lambda(\cdot)$ and then outline how to get the expression in (2.14). Elementary but tedious computations give

$$I'(\Lambda'(\lambda)) = \lambda. \tag{5.7}$$

It can be shown that Λ' is continuous and strictly increasing. Therefore, I' is defined on the whole real line by (5.7). By Theorem 26.5 in [12] (page 258), I must be Λ^* plus a constant. But $I(0) = 0 = \Lambda^*(0)$, so $I = \Lambda^*$. This proves that I given by (2.14) is the convex dual of Λ and hence the rate function. Since Λ' is continuous and strictly increasing, I' must be strictly increasing from (5.7) and hence I must be strictly convex. This completes the proof of Theorem 2.2. Corollary 2.2 comes as a special case of (2.14).

For the reader's benefit, here is an indication of how the expression in (2.14) is derived non-rigorously. We first approximate the integral in $\Lambda(\cdot)$ by a Riemann sum.

$$\Lambda(\lambda) = \lim_{\delta \to 0} \sum_k \tilde{\Lambda}_k^\delta(\lambda)$$

where

$$\tilde{\Lambda}_k^\delta(\lambda) := \begin{cases} \delta\gamma\big(Br_{1-\Phi_{\kappa_2 t}(k\delta)}(\lambda)\big) & \text{for } k > 0 \\ \delta\gamma\big(Br_{\Phi_{\kappa_2 t}(k\delta)}(-\lambda)\big) & \text{for } k \leq 0. \end{cases} \tag{5.8}$$

The function $Br_p(\lambda)$ denotes the log moment generating function for Bernoulli random variables as defined in section 2.3. Observe that the summands (5.8) are a composition of two functions. Using the definition of convex dual, it is easy to prove the identity

$$(f \circ g)^*(x) = f'(g(\lambda))g^*(g'(\lambda)) + f^*(f'(g(\lambda))) \tag{5.9}$$

where λ is such that

$$x = f'(g(\lambda)) \cdot g'(\lambda). \tag{5.10}$$

We use this identity to get the convex dual of the summands (5.8).

For small δ

$$\Lambda(\lambda) \approx \sum_{|k| \leq [1/\delta]} \tilde{\Lambda}_k^\delta(\lambda).$$

The convex dual of the sum is then given as an infimal convolution.

$$\Lambda^*(x) \approx \left(\sum_{|k|\le[1/\delta]} \tilde{\Lambda}_k^\delta \right)^* (x) = \inf_{\sum_{|k|\le[1/\delta]} x_k = x} \sum_{|k|\le[1/\delta]} \left(\tilde{\Lambda}_k^\delta\right)^* (x_k)$$

now using (5.9) we get

$$\begin{aligned} = \inf_{\sum\limits_{|k|\le[1/\delta]} x_k = x} \Bigg\{ \delta \sum_{k=1}^{[1/\delta]} \Big[& \gamma' \left(Br_{1-\Phi_{\kappa_2 t}(k\delta)}(\lambda_k) \right) Br^*_{1-\Phi_{\kappa_2 t}(k\delta)}(Br'_{1-\Phi_{\kappa_2 t}(k\delta)}(\lambda_k)) \\ & + \gamma^* \left(\gamma'(Br_{1-\Phi_{\kappa_2 t}(k\delta)}(\lambda_k)) \right) \Big] \\ + \delta \sum_{k=-[1/\delta]}^{0} \Big[& \gamma' \left(Br_{\Phi_{\kappa_2 t}(k\delta)}(-\lambda_k) \right) Br^*_{\Phi_{\kappa_2 t}(k\delta)}(Br'_{\Phi_{\kappa_2 t}(k\delta)}(-\lambda_k)) \\ & + \gamma^* \left(\gamma'(Br_{\Phi_{\kappa_2 t}(k\delta)}(-\lambda_k)) \right) \Big] \Bigg\} \end{aligned} \tag{5.11}$$

where

$$x_k = \begin{cases} \delta\gamma' \left(Br_{1-\Phi_{\kappa_2 t}(k\delta)}(\lambda_k) \right) Br'_{1-\Phi_{\kappa_2 t}(k\delta)}(\lambda_k) & \text{for } k > 0 \\ -\delta\gamma' \left(Br_{\Phi_{\kappa_2 t}(k\delta)}(-\lambda_k) \right) Br'_{\Phi_{\kappa_2 t}(k\delta)}(-\lambda_k) & \text{for } k \le 0. \end{cases}$$

Note that

$$Br_{1-\Phi_{\kappa_2 t(y)}}(\lambda) = Z_\lambda(y) \text{ for } y > 0 \ , \ Br_{\Phi_{\kappa_2 t(y)}}(-\lambda) = Z_\lambda(y) \text{ for } y \le 0 \tag{5.12}$$

$$Br'_{1-\Phi_{\kappa_2 t(y)}}(\lambda) = 1 - F_\lambda(y) \text{ for } y > 0 \ , \ Br'_{\Phi_{\kappa_2 t(y)}}(-\lambda) = F_\lambda(y) \text{ for } y \le 0. \tag{5.13}$$

Taking the limit as $\delta \to 0$ of (5.11) we get the constrained variational problem

$$\begin{aligned} \Lambda^*(x) = \inf_{\{\lambda : \int\limits_{-\infty}^{\infty} \frac{\partial}{\partial \lambda(y)} \left[\gamma \left(Z_{\lambda(y)}(y) \right) \right] dy = x\}} & \Bigg[\int_{-\infty}^{\infty} \gamma^* \left\{ \gamma'(Z_{\lambda(y)}(y)) \right\} dy \\ & + \int_{-\infty}^{\infty} \gamma'(Z_{\lambda(y)}(y)) Br^*_{\Phi_{\kappa_2 t(y)}}(F_{\lambda(y)}(y)) dy \Bigg]. \end{aligned}$$

Solving the above variational problem using standard functional analysis techniques we get $\lambda(y) \equiv \alpha(x)$ minimizes the above functional and $\alpha(x)$ satisfies the constraint (2.11). $\square$

Proof of Theorem 2.3. Let $Y_n(t) = Y_n(t,0)$. Under the assumption $\eta_0^n(m) \sim Poisson(\rho)$ we can show that $\{n^{-1/2}Y_n(\cdot)\}$ satisfies the large deviation principle in $D_{\mathbb{R}}[0,\infty)$.

Fix k time points $0 \le t_1 < t_2 < \cdots < t_k$. Define the k-vectors with $0,1$ entries by

$$\vec{F}_{m,j} = \big(\mathbf{1}\{X_{m,j}(nt_1) \le [nvt_1]\}, \dots, \mathbf{1}\{X_{m,j}(nt_k) \le [nvt_k]\}\big)$$

and

$$\vec{G}_{m,j} = \big(\mathbf{1}\{X_{m,j}(nt_1) > [nvt_1]\}, \dots, \mathbf{1}\{X_{m,j}(nt_k) > [nvt_k]\}\big).$$

Then

$$\big(Y_n(t_1), \dots, Y_n(t_k)\big) = \sum_{m=1}^{\infty} \sum_{j=1}^{\eta_0^n(m)} \vec{F}_{m,j} - \sum_{m=-\infty}^{0} \sum_{j=1}^{\eta_0^n(m)} \vec{G}_{m,j}.$$

Let $\vec{u} := (u^{(1)}, \dots, u^{(k)}) \in \{0,1\}^k$ be a k-vector with $\vec{u} \ne \vec{0}$. Define

$$N_n^1(\vec{u}) := \sum_{m=1}^{\infty} \sum_{j=1}^{\eta_0^n(m)} \mathbf{1}\{\vec{F}_{m,j} = \vec{u}\}$$

and

$$N_n^2(\vec{u}) := \sum_{m=-\infty}^{0} \sum_{j=1}^{\eta_0^n(m)} \mathbf{1}\{\vec{G}_{m,j} = \vec{u}\}.$$

If $\eta_0^n(\cdot)$ are i.i.d. Poisson(ρ) random variables, then $N_n^1(\vec{u})$ is a Poisson random variable with rate

$$\sum_{m=1}^{\infty} \rho P\Big(\bigcap_{i=1}^{k} C_{m,i}^{u^{(i)}}\Big) < \infty,$$

where $C_{m,i}^1 = \{X(nt_i) \le [nvt_i] - m\}$, $C_{m,i}^0 = \{X(nt_i) > [nvt_i] - m\}$, and $N_n^2(\vec{u})$ is a Possion random variable with rate

$$\sum_{m=-\infty}^{0} \rho P\Big(\bigcap_{i=1}^{k} D_{m,i}^{u^{(i)}}\Big) < \infty,$$

where $D_{m,i}^1 = \{X(nt_i) > [nvt_i] - m\}$, $D_{m,i}^0 = \{X(nt_i) \le [nvt_i] - m\}$. We use the bounds (3.1) and (3.2) on the large deviations of random walks to justify the rates being finite.

We can write

$$(Y_n(t_1), \dots, Y_n(t_k)) = \sum_{\vec{u} \ne \vec{0}} (N_n^1(\vec{u}) - N_n^2(\vec{u}))\vec{u}.$$

Let $\vec{u}_j = (u_j^{(1)}, \dots, u_j^{(k)}), j = 1, \dots, 2^k - 1$ denote the $\{0,1\}$-valued k-vectors, excluding the zero vector. By the contraction principle (Theorem 4.2.1 in [5]) we can conclude that $(Y_n(t_1), \dots, Y_n(t_k))$ satisfies the LDP with good rate function given by

$$I_{t_1,\dots,t_k}(\vec{x}) := \inf J(y_1, \dots, y_{2^k-1}, z_1, \dots, z_{2^k-1})$$

for any $\vec{x} = (x_1, \ldots, x_k)$. The inf here is taken over the set $\{(y_1, \ldots, y_{2^k-1}, z_1, \ldots, z_{2^k-1}) : \vec{x} = \sum_{\vec{u}_j \neq \vec{0}} (y_j - z_j)\vec{u}_j\}$. $J(y_1, \ldots, y_{2^k-1}, z_1, \ldots, z_{2^k-1})$ is the good rate function for the sequence of vectors $\{(N_n^1(\vec{u_1}), \ldots, N_n^1(\vec{u}_{2^k-1}), N_n^2(\vec{u_1}), \ldots, N_n^2(\vec{u}_{2^k-1}))\}_n$ of $2^{k+1} - 2$ independent Poisson random variables. Define for $x \geq 0$,

$$C_{x,j}^1 = \{B(\kappa_2 t_j) \leq -x\},\ C_{x,j}^0 = \{B(\kappa_2 t_j) > -x\},$$

$$D_{x,j}^1 = \{B(\kappa_2 t_j) > x\} \text{ and } D_{x,j}^0 = \{B(\kappa_2 t_j) \leq x\},$$

where $B(\cdot)$ is standard Brownian motion.

$$J(y_1, \ldots, y_{2^k-1}, z_1, \ldots, z_{2^k-1}) = \sum_{i=1}^{2^k-1} \left[y_i \log \frac{y_i}{\alpha_i} + z_i \log \frac{z_i}{\beta_i} - \alpha_i(\frac{y_i}{\alpha_i} - 1) - \beta_i(\frac{z_i}{\beta_i} - 1) \right]$$

where

$$\alpha_i = \lim_{n\to\infty} \frac{1}{\sqrt{n}} E N_n^1(\vec{u_i}) = \rho \int_0^\infty P\left(\bigcap_{j=1}^k C_{x,j}^{u_i(j)} \right) dx$$

and

$$\beta_i = \lim_{n\to\infty} \frac{1}{\sqrt{n}} E N_n^2(\vec{u_i}) = \rho \int_0^\infty P\left(\bigcap_{j=1}^k D_{x,j}^{u_i(j)} \right) dx.$$

We now apply Theorem 4.30 in [7] to $\{n^{-1/2} Y_n(\cdot)\}$. This gives us the large deviation principle for $\{n^{-1/2} Y_n(\cdot)\}$ in $D_{\mathbb{R}}[0, \infty)$ with good rate function

$$I(x) = \sup_{\{t_i\}} I_{t_1,\ldots,t_m}(x(t_1), \ldots, x(t_m)). \tag{5.14}$$

□

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